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EDITORS NOTE

This issue has been devoted to papers presented at the 2nd International Conference on Operational Research in the session entitled "Evaluation of Inventory Control". This was an invited review session and was organised by Dr. E. Naddor, of the Department of Industrial Engineering, Johns Hopkins University, Baltimore, Maryland. Dr. Naddor's introductory paper appears in full in the published Proceedings of the Conference (English Universities Press Ltd.), but due to space restrictions in the Proceedings the papers contributing to the session were only published in abstract.

Study of the Optimum Equipment and Utilization of an Underground Gas Reservoir of Limited Capacity*

R. FORT

I

THE fluctuations of demand in gas-supply networks, which increase considerably with the development of gas heating, make systematic research into the minimum cost of peak-load gas supply imperative.

It is now usual in France to present the throughput that is likely during a future year by the curve known as the *S*-curve giving, as a function of each possible value of daily throughput, the mean number of days on which this throughput will be reached or exceeded (Figure 1).

Studies have revealed that three basic concepts are involved in the determination of the most economical techniques and of their respective importance, namely, the marginal operating cost, most of which is connected with the raw materials (coal, fuel, oil, LPG, natural gas), the cost of the equipment for storing the raw material, expressed per unit quantity (m^3 or therm), and the annual cost (financial and personnel expenses) of the various items of equipment, the magnitude of which depends on the rate of output (m^3 per day or therm per day) and which cover all the operations from withdrawal of the gas from storage up to its consumption.

* Paper presented at the Second International Conference on Operational Research, held at Aix-en-Provence, 5th to 10th September 1960.

Underground storage has proved to be a very economic method of satisfying heating peak demands because its "material" cost is not very different from that of basic gas supply (the two methods differ in the costs of compression and treatment of the gas), because the cost of the equipment per m^3 of gas stored is low, and because the yearly costs of transporting the withdrawn gas to the distribution networks are far lower than those of producing plants. This is assuming, of course, that gas is stored for an immediate distribution, otherwise there would also be the yearly costs of producing equipment.

Gaz de France found it necessary to carry out research in the Paris area, where the quantity of gas used for heating is now considerable. These works could only be carried out in an aquifer structure, since no oil or gas field had been exploited in this area. This undertaking met with success and ended in the discovery of a reservoir at Beynes.

The economic studies carried out to determine the best way of utilizing this reservoir revealed the far-reaching effect of some of its characteristics. One of these characteristics is the water drive,* the effect of which is the more pronounced, with the flat structure and relatively low total volume, as is the case at Beynes.

This paper describes the method used to determine the optimum equipment for such a reservoir. We shall give also the necessary indications to apply this method to the case of a reservoir with a very low water-drive but also with a limited capacity.

2

The gas-producing equipments or the associated transmitted equipments (i.e. transporting gas fit for direct distribution) should be designed for a maximum throughput defined, in France, as that which is not likely to be exceeded in more than two years in 100 years. This point (E_0 in Figure 1) limits the peak of the *S*-curve.

In the usual types of equipment the direct (marginal) cost varies little with the operating regime (we then speak of the proportional cost), so that the programme of optimum utilization consists of bringing units into operation in order of magnitude of direct costs, subject to eventual contractual duties concerning the raw material purchases (transported gas, propane, etc.).

For one given equipment which, on the basis of this programme, will have to ensure throughput between the values E_m and E_M, the mean yearly volume provided will be represented by the hatched area in Figure 1. In fact, since different years will have different degrees of cold, the actual values will fluctuate round the mean value. Statistically, this mean value S is defined by

$$S = \int S_i p_i \, dS_i \quad \text{with} \quad \int p_i \, dS_i = 1 \qquad \text{(A)}$$

* For an explanation of the technical terms, the references should be consulted.

where $p_i\,dS_i$ is the probability of having an area included between S_i and S_i+dS_i.

In choosing the equipment, we work on the basis that the yearly expenses in a throughput range included between E_i and E_i+dE_i can be expressed in the form $[A+Bn(E_i)]\,dE_i$. Under these conditions the optimum limit of application

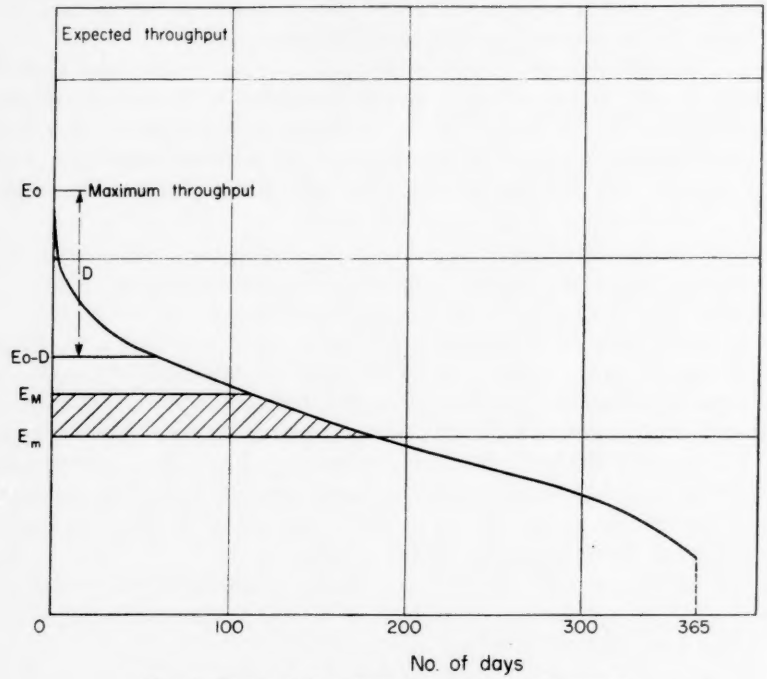

FIG. 1. *Curve of throughput frequencies or "S-curve"*
abscissa = No. of days
(Expected throughput; maximum throughput)

of the two techniques k and p is defined by equality of the yearly costs in the limiting range, i.e.

$$A_k+B_k n_{\text{lim}} = A_p+B_p n_{\text{lim}} \tag{1}$$

where

$$n_{\text{lim}} = \frac{A_p-A_k}{B_k-B_p}. \tag{B}$$

By inserting the value of n_{lim} in Figure 1, we get the throughput limit.

The costs of stocking the raw material do not generally affect the expression of this result very much, except in cases where the corresponding equipments are relatively expensive, e.g. propane reservoirs. It is usual, therefore, to maintain stocks that are adequate for any eventuality, whereas, on the contrary,

supplies of propane are kept within tight limits so that the storage capacity can be restricted to that which is strictly necessary.

In designing an underground reservoir, the problem is quite different. The equipment has to be designed for given in-put (injection) and out-put (withdrawal) rates. The storable volume is deducted from integrating over the low throughputs (off-peak) season, the available gas surpluses being cut off at the injection rate value (rate of compressors). The marginal cost of the gas to be withdrawn can be simultaneously deduced therefrom.

If we adopted the rule applying to producing equipment and used the throughput range from the time when the comparison of the marginal costs prescribes this use, we should be led to withdraw a volume V of variable magnitude depending on the severity of the winter, the mean value being represented by the area S defined above; we will denote the maximum value by V_M.

It might be considered, therefore, that any withdrawal rate I should correspond to an injection rate I, so that the volume $V_M(D)$ can be restored each year.

However, in the case of a reservoir of limited volume, the marginal cost of the equipment, which is normally rather low, becomes very high if we attempt to withdraw too great a percentage of the total capacity of the reservoir,* and so it appears profitable to withdraw from the underground reservoir only after having utilized the other producing plants, the marginal cost of which is higher.

This solution is the more interesting moreover, since it is so easy to design the underground store to meet extreme peak demands in very severe winters and to come nearer to the solution of destocking as a function of the marginal costs, so far as the severity of the winter permits this.

It was on the basis of operating conditions conceived in this way that we tried to find the answer to the problem of the optimum equipment. We will show later how the operating instructions are formulated.

3

Assuming that the reservoir be equipped to provide a withdrawal rate D, the throughput range for the severest winter is then, by working hypothesis, utilized between the values $E_0 - D$ and E_0 (Figure 1), thus corresponding to the "extreme peak" regime.

It was necessary, therefore, first of all to determine the volume corresponding to this throughput range that is unlikely to be exceeded in more than two years in 100. We did this assuming the validity of the classical theory of linear variation in the throughput as a function of the temperature $E = A - Gt$, the coefficient G being the "throughput gradient". The ordinate $E_0 - D$ then corresponds

* The portion of the stock that is not used is called cushion gas. Thus $t_{\text{total}} = V_{\text{cushion}} + V_M$. We will indicate later the part played by the cushion gas.

to a well-defined temperature t_r and the volume to be withdrawn is then the product $G.N, N$ being the number of "degree-days" below t_r. This number is $\Sigma(t_r - t_i)$, where t_i represents any temperature less than t_r, the sum Σ extending over all days for which the temperature is less than t_r.

For a given year for which the temperatures are known, N can be determined either for the whole winter or for any period between a fixed date and the end of the winter.*

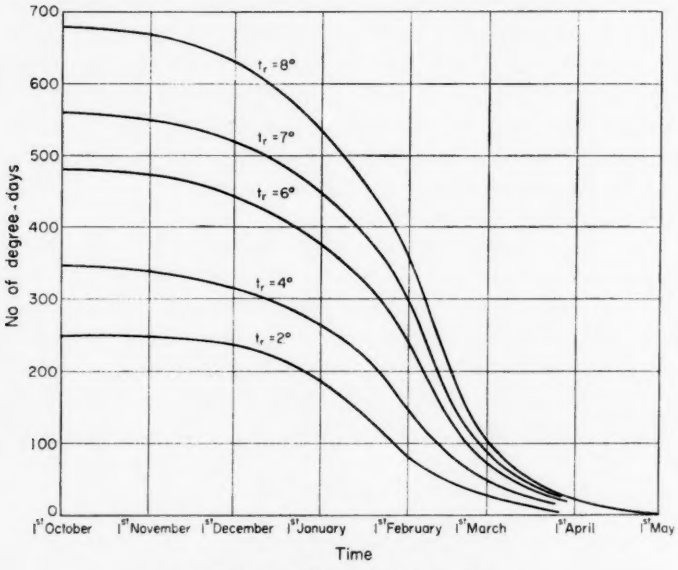

FIG. 2. *Evolution of safety stock*
abscissa = Time (1st October—1st May)
ordinate = No. of degree-days

From temperature statistics covering 50 years, we determined the corresponding values of N relative to a single period and made a statistical analysis thereof. This research, carried out by the Operational Research Group of Gaz de France, showed that after transposing the variable a log-normal distribution was obtained. It was then easy to determine the value N_M that is unlikely to be exceeded more than 2 years in 100. We thus obtained Figure 2 representing the value N_M for different values of t_r, i.e. of D, as a function of the output rate in the period of winter being considered. By definition, $V_M = GN_M$.

* The notion of the end of the winter may appear subjective. It is in fact the time beyond which no temperature below t_r is found. The concept will become tenuous only for a temperature t_r fairly high as might be observed in summer.

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To get a better understanding of the object of the economic studies, we should now recall some technical aspects that bear on the costs.

The underground reservoir is a water-bearing stratum, sand or porous rock in the shape of a dome and surmounted by an impermeable layer. The gas spreads below the impermeable layer and displaces the water in the stratum. If equilibrium were reached, the water and the gas would be separated by a horizontal plane, the volume of stored gas being defined by $v\delta^P$, where v represents the volume of the structure above the level of the water, δ the percentage of void (porosity) and P the hydrostatic pressure at the level of the surface of the water.

During injection and withdrawal, however, the water does not disappear or return immediately to restore the equilibrium. There is a time lag, the duration of which varies with the water drive, a characteristic that depends on the permeability and volume of the rocks in which the water can circulate, whether in the reservoir-stratum or in the strata communicating therewith. The proximity of outcrops is also of great importance.

Thus each underground reservoir is a special case, the two extremes corresponding on the one hand to a very active water drive, the water level being displaced immediately to restore the hydrostatic pressure, and on the other hand, to virtually zero water drive,* the reservoir acting in effect as a closed vessel in which the gas pressure evolves according to the corrected Mariotte law ($PV = ZRT$), as a function of the injections or the withdrawals.

The value of the withdrawal or injection rate of a well can be expressed in the form:

$$d = k[h(P^2 - P'^2)]^n \tag{2}$$

where k is a coefficient dependent in particular on the permeability of the rock forming the reservoir, h represents the depth of the borehole in the porous zone, P and P' are the gas pressures at the foot of the borehole and at the limit of its zone of action, and n is a characteristic coefficient of the rock.

In a reservoir with a very active water drive—like the Beynes reservoir, another factor is very important, viz. withdrawal of gas creates a reduced pressure immediately above the water level which then becomes deformed to produce water coning. At the limit, the height of this cone is given by the hydrostatic relation $h' = g(P - P')$. Thus the perforations should be made as high as possible in the stratum, and the pressure drop during withdrawal should be limited to h'/g (Figure 3).

For a given level H of the surface of the water, by making the approximation $P^2 - P'^2 \sim 2P(P - P')$ (because of the small difference between P and P') it is easy to show that the optimum lowest level of the perforations is $H/2$.

* In this case, moreover, in order to fill the reservoir, water has to be removed from the outer zones of the structure at the same time as gas is injected.

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Resuming our economic study, therefore, we see that the height of the water level that has to be taken into consideration in order to design the withdrawal equipment suitable for ensuring a guaranteed rate D, is the highest water level which settles at the end of the withdrawal period. The extreme value of the volume withdrawn being V_M,* it suffices to find the relation between the height of the water level H and V_M, for which it is simply necessary to know fairly well the shape of the structure and the local porosity of the stratum.

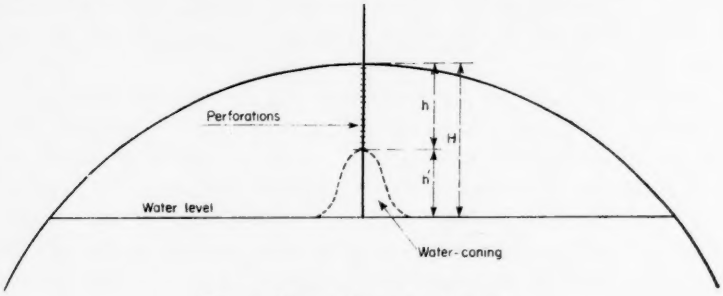

FIG. 3. *Phenomenon of water-coning*
Perforations
Water level
Water-coning

Taking the argument as a whole, it will be seen that in order to be sure of being able to withdraw at any time a rate D, according to an operating programme providing an extreme-peak for the severest winter, we know therefrom the volume $V_M(D)$ that it may then be necessary to withdraw (values given by Figure 2 on 1st October, t_r being a function of D). We derive therefrom the minimum value of H, hence the optimum perforation height $H/2$ and, by means of equation (2), the *guaranteed* withdrawal rate from a well d_G. The number of wells required is thus the whole number immediately greater than D/d_G. The minimum pressure obtainable at the top of the well P_m is found in the same way.

In the case of a reservoir with zero water drive, according to the corrected Mariotte law we can write:

$$Z(P_g\, P_g) = Z(P_0)\, P_0 \frac{V_{\text{total}}}{V_{\text{total}} - V_M}$$

which represents a relation between P_g and V_M (P_g is the final pressure inside the stratum, P_0 the initial pressure before any withdrawal).

Equation (2) also gives the rate of a well if we replace P by P_g: h is then the total height of the stratum. It is thus possible to find, as a function of P', the number of wells required to guarantee the rate D. Similarly, as a function of P',

* The problem is actually simpler than this, because the rate D required for the *coldest* temperature need not be guaranteed *all winter*. Experience shows, however, that this temperature can be found up to the end of February and, as will be seen from Figure 2, it is sufficient to decrease the volume V_M slightly to get a correct estimate.

we can find the size of the pipelines and recompression equipment necessary in order to discharge the rate D to the distribution network. Thus, there is an optimum value of P' that ensures that the yearly costs of the whole system are minimized. We thus get simultaneously this value P_m, the optimum number of wells, and in addition, the structure of the discharge system of the withdrawn gas.

9

On the basis of the data derived above for the two special cases considered, we are now able to determine the optimum equipment according to the classical methods.

We determine first of all the equipment required for withdrawal, i.e., the diameter of the pipelines connecting the reservoir to the distribution network, and the size of the recompression station required in order to ensure that the rate D of available withdrawn gas is discharged at the pressure P_m. In the case of a reservoir with zero water drive, this calculation has already been done, since it had to be allowed for in studying the partial optimum. We add then the equipment that may be necessary to treat the withdrawn gas (eliminate impurities formed during storage, dessicate the gas, etc.). We then study the optimum supplementary equipment required to ensure injection of the volume $V_M(D)$. This is generally supplementary equipment, since the rate of injection of gas is normally less than that of withdrawal, because of the shape of the graph in Figure 1, so the withdrawal pipes are often used to inject gas, unless the stored gas comes from elsewhere than from the area being served. Finally, we add the equipment for treating the gas during injection.

It is thus possible, for discrete values of D, to find the yearly fixed costs F of the whole of the equipment, and the marginal cost B_s of the gas withdrawn from the reservoir. In the case where the reservoir equipment does not include a recompression station, the discrete values of D are selected so that they correspond to the standard diameters of the steel pipes that will link the reservoir to the distribution network.

Equation (1), used in designing the producing equipment, will then have to be transposed to determine the limit of the development of the underground reservoir beyond which it is more advantageous to employ a production unit than to increase the withdrawal rate of the underground storage.

For this we have to compare the yearly costs of the two rival types of equipment in the marginal range of the throughput rate.

In the case of the underground storage, the fixed annual charges for this marginal range are obtained by calculating the value of dF/dD.*

* It may appear unusual to take the derivative of a function that becomes discontinuous as a result of standardization of the pipe diameters. This is common practice, however, and the only disadvantage with this method is that, once the theoretical optimum has been determined, it is necessary to repeat the calculation for the two practical solutions bracketing the optimum.

In the case of the production units the fixed charges are proportional to the rate; in effect, underground stores have no advantage except in the zones of high consumption, where production is ensured by high-output units, thus corresponding to a technical maximum, for which the concept "marginal" is meaningless.

With regard to the direct charges, here the problem is more complex, and we shall, therefore, examine more closely the choice of the operating regime.

Two cases should be considered:

(a) If the stored gas has a marginal cost B_s greater than that of the produced gas B_p,* the operating regime is quite clear cut.

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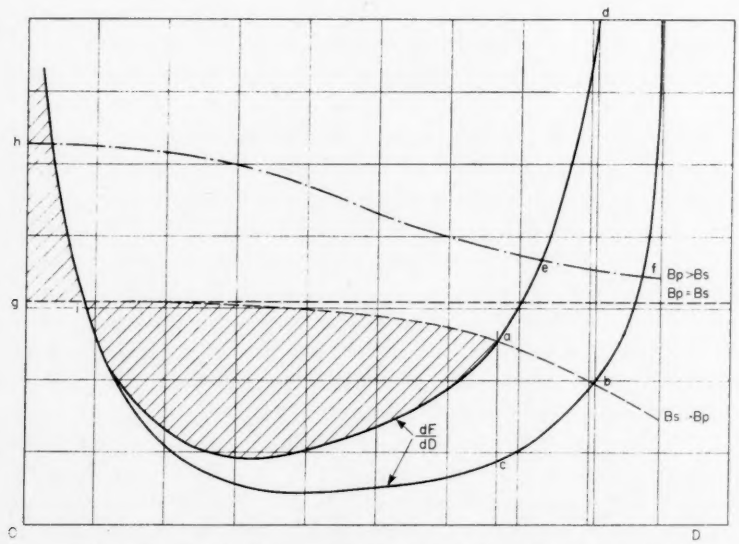

FIG. 4. *Graphical solution of the problem of determining the optimum*
ordinate = Annual cost per m³/day, abscissa = m³/day

In this case, in fact, economic considerations oblige us to adopt systematically the extreme-peak operating regime, which up to now we have not considered using except where necessary for safety reasons.

With such a regime, the volume used as the mean in the marginal range is that given by the *S*-curve (Figure 1), i.e. $n(E_0 - D)$.

* This would be the case, for example, if all the production were carried out by the same technique, the stored gas being taken from this production, the difference in the costs then being made up of the costs of treating and compressing the gas.

The difference in the proportional charges will then, to the detriment of underground storage, assume the following value:

$$(B_s - B_p).n(E_0 - D). \tag{D}$$

The optimum withdrawal rate from the underground reservoir is then given in this case by the equation:

$$\frac{dF}{dH} = A_p - (B_s - B_p)n(E_0 - D). \tag{3}$$

This equation can be resolved easily by means of a graph (Figure 4). In this graph, we have represented dF/dD by two solid-line curves. These two curves indicate the uncertainty inherent in the actual expenses of equipment to be provided for; these expenses depend on the exact geological characteristics of the reservoir-stratum of which, at the time the research was carried out, we had only partial information (obtained from a small number of borings made at the time).

The fixed charges Ap are represented by the horizontal broken line. The second term in equation (3) is represented by the broken-line curve below this, which will be seen to be analogous with the peak in Figure 1.

(b) If, on the other hand, the stored gas has a marginal cost lower than the proportional cost of the produced gas, the operating regime is different. We then have to define it very accurately.

We have seen in effect that, in order that the consumers be served with a failure of risk of 2 per cent, we have to keep in stock and available a volume of gas at least equal to that corresponding to the guaranteed rate D, i.e. that shown in Figure 2 for the value of t_r corresponding to this rate.

The control of the system being represented by the change in the volume of stored gas as a function of time, we insert this curve in a (V, t)-diagram, the original of V being the cushion volume.* This curve we shall call *alarm curve.*

For a better illustration of what takes place, it is also possible, on the basis of the statistical laws of temperature, to determine the minimum volume that is withdrawn from storage at the extreme peak in a particularly mild winter, which we have included in Figure 5. Here also, we chose figures that were not likely to be exceeded more than 2 per cent of the years. Consequently, if we adopt an extreme-peak regime the date volume of the stored gas has 96 chances in 100 of lying between the two curves.

* In reality, bearing in mind the footnote on p. 4 the volume of the cushion gas can be represented by a curve starting with a straight-line section and having an inflexion point as soon as there is a fall in the minimum temperature that is likely to be observed later (i.e. after the end of February). In this case, the curve retained from Figure 2 is moved vertically until it forms a tangent to this curve of evolution of the cushion gas.

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It will be seen immediately that, if the point representative of the stock level lies below the alarm curve,* the risk of the maximum consumer demand not being fulfilled will exceed 2 per cent. For safety reasons, it is vital to reduce this risk as much as possible, and consequently, the withdrawal of gas should be reduced to its lowest value, corresponding to extreme-peak operation.

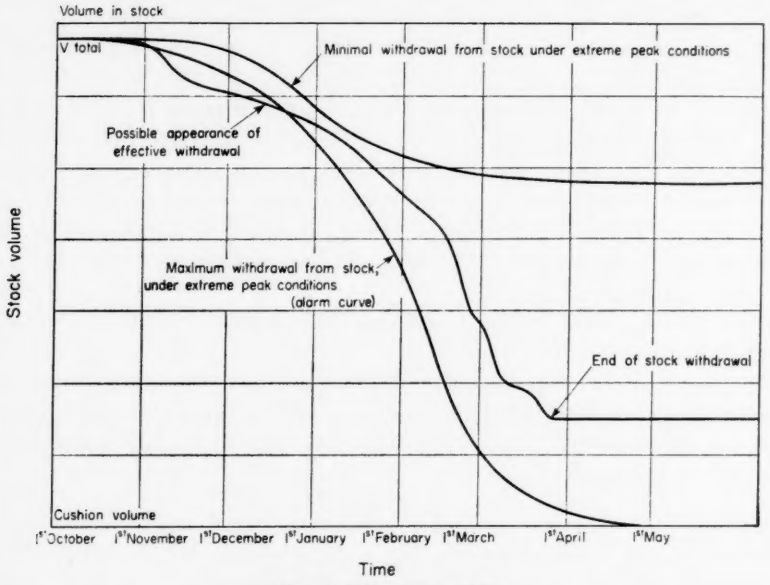

FIG. 5. *Operating regime*
ordinate = Stock volume
abscissa = Time (1st October—1st May)

On the other hand, if the representative point lies above the alarm curve, in terms of the safety stock there is available a surplus stock V_e that can be used to improve the economics of operation.

Three conditions will guide us in choosing the operating system:

(1) The stock can be used immediately with a rate dD; this use allows the reducing of the proportional charges of produced gas at which the withdrawal will be substituted. We know exactly the evolution of the proportional cost of produced gas as a function of d, or $C_i(d)$.

(2) It is also possible to postpone utilization of the stock with a rate $d < D$. In this case, however, on account of the risk involved in making the throughputs during the rest of the winter it is no longer possible to determine exactly the

* This situation should only arise very exceptionally (risk of 2 per cent). It may be brought about, however, as a result of incidents during the stock period preventing the stock from reaching its maximum level at the start of the winter.

value of substitution. Instead, we have to determine the "mathematical expectation of marginal return",* by utilizing the probability of various possible throughput models and the corresponding return, account being taken of the extra interest incurred through postponement of utilization of the stock, i.e. we calculate the "actual expectation". This value depends both on the size of the surplus stock and on the time at which it is available: we shall denote it, therefore, by $E_v(V_e, \theta)$.

(3) The surplus stock can also be retained up to the end of the winter. In this case, since the reservoir has a small capacity and thus has to be filled completely during the summer, such a solution will reduce the volume that has to be injected back during the summer, and thus the marginal costs of restocking. Since the throughput provisions are best known in summer, it is possible to determine almost exactly this marginal cost as a function of the volume to be restocked, and the period over which the ultimate volume will be spread. It is thus possible to determine an "actual cost of marginal restocking" $C_r(V_s)$ for any value of the stock V_s.

The choice of operation thus depends on the comparison of these three values $C_i(d)$, $E_v(V_e, \theta)$ and $C_r(V_s)$.

If we compare first $C_i(d)$ with $C_r(V_s)$, which is easiest, it is clear that there is no point in withdrawing gas the restocking of which would cost more in actual value than the marginal cost of gas production. Withdrawal of gas is, therefore, unjustifiable unless:

$$C_i(d) \geqslant C_r(V_s). \tag{E}$$

Since $C_i(d)$ is a diminishing function of d, withdrawal will be justified if this inequality is valid for $C_i(0)$, and the rate d to be adopted will be such as to make the costs equal.

This leads us to define how the value $E_v(V_e, \theta)$ should be calculated: it should not be thought that the withdrawal of V_e is obligatory, the possibility of maintaining it in stock should also be admitted. In the calculation, any gas left in stock will, therefore, be priced at the actual cost of restocking.

We then compare the two values $C_i(d)$ and $E_v(V_e, \theta)$. In principle, this single comparison will suffice since, according to what was said in the previous paragraph $E_v(V_e, \theta)$ allows for the possibility of maintenance in stock implying that:

$$E_v(V_e, t) \geqslant C_r(V_s). \tag{F}$$

The inequality of $C_i(d) \geqslant E_v(V_e, \theta)$ will, therefore, show whether withdrawal from stock is justified (inequality valid for $d = 0$) and what limit d of the withdrawal rate should be adopted (the value of d leading to equality, cut off if necessary at D).

* Concept introduced by M. Massé in the study of the operation of hydraulic electricity reservoirs; cf. Massé "Les ressources et la régulation de l'avenir" (Hermann, 1946).

Clearly, in defining such an operating regime it is not possible to give a mathematical formulation of the gains so accruing with regard to the direct charges. To determine these gains it is necessary:

(a) to know the exact structure of the gas supply organization in the plants of which the underground reservoir is incorporated;

(b) to determine the mean consequences of the adoption of this method of operation with the different withdrawal rates envisaged. Since it is impossible to express the risk in strictly mathematical form, this result will be obtained by the Monte Carlo method.

However, in order to exemplify the consequences of the adoption of such a method of operation, we give a possible description of the curve of the differences in the direct charges. This description features the possibility of utilizing very fully the ranges of low total withdrawal rate (thanks to a very cheap fitting-up of the withdrawal wells) and the low coefficient of utilization of the extreme ranges of withdrawal.

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Thus Figure 4 gives, in a precise manner for the case $B_s > B_p$, in an approximate manner for $B_s < B_p$,

(a) the optimum withdrawal rate, and

(b) the advantage of the underground reservoir over the classical methods (difference in the hatched areas minus the difference in the absolute fixed costs, i.e. the costs that arise irrespective of the size of the system). The hatched areas correspond to the case $B_s > B_p$ with the most unfavourable geological conditions. By the juxtaposition on this graph of curves corresponding to various price conditions and extreme geological conditions, it is possible to draw a number of conclusions, some general and some connected with the type of reservoir being studied.

The dF/dD curve falls initially, corresponding to the fall in the cost of transport via the pipeline (link between the reservoir and the distribution network). After passing through a minimum that is more or less flat, the curve rises again fairly rapidly, the increase in the costs here being due to the increase in the number of wells. The degree of spreading of the central zone is a function solely of the total capacity of the reservoir, and the rise in the curve indicates that the resources of the reservoir are exhausted.

In this way, we can define more closely the expression "limited volume" used to characterize the underground reservoir. If the reservoir had a surplus volume, the dF/dD curve would be more spread out. In addition, underground storage would then have to be compared, not with an extension of peak-load gas supply, but with an extension of basic gas supply, and the dotted curve would turn downwards, since the marginal cost of underground storage B_s is equal to the marginal cost of basic gas supply plus the costs of treatment of the gas and of

compression for injecting into the reservoir. Under these conditions, the economic withdrawal volume D_e (point of intersection of the two curves) would lie in the central zone of dF/dD. An increase in the sales of gas modifying the elongation of the *S*-curve would lead to a practically proportional increase in D_e. In such a case, the withdrawal rate is primarily a function of the throughput characteristics rather than of the geological structure. The reservoir is, therefore, ample. If on the other hand, as in Figure 4, the point of intersection lies in the rising section of the curve, the value $B_p - B_s$ is besides generally positive, since here the withdrawn gas is compared with peak-load produced gas, which has a higher proportional cost than the basic gas used during injection of gas into the reservoir. In this case, modification of the throughput characteristics only very slightly affects the economic withdrawal rate D_e. The optimum rate is then essentially connected with the structure of the reservoir and only slightly with the throughput characteristics. This is a characteristic of a reservoir of limited volume.

It also appears that the optimum withdrawal rate defined by the ordinate of the point of intersection of the curves—a point which, depending on the precise conditions at the time, will be a, b, e or f—varies in fact relatively little with these conditions. Moreover, although too much importance should not be attached to this conclusion, which depends on the special characteristics of the example being studied—the optimum value is influenced more by the geological conditions than by the price of the available gas volume. Thus, modifications to the supply system are not likely to call back the choice of the equipment in question again.

On the other hand the advantage of underground storage is influenced a great deal by the prices of the various gases, the extra cost being represented in the example mentioned by one or other of the areas *gaeh* or *gbfh*. This advantage is also greatly dependent on the geological characteristics, the difference between the two extreme hypotheses being represented by the curvilinear triangle *abi* or *efi*. For all that, it should be noted that this difference only corresponds to the risk affecting a structure that has already been partly recognized, from which it may be concluded that, for storage in a water-bearing stratum, it is vital to carry out progressive and systematic research so that no expense is involved unless the structure offers promise of adequate success.

The problem of choosing the optimum rate has still another bearing that is connected with the need to develop some equipments before the risk inherent to the geological data has been totally eliminated. If all the expenses had to be incurred before this risk has been eliminated, on selecting (a), one would risk losing the economic advantage represented by the curvilinear triangle *abc*. On the other hand, on selecting (b) one would risk having an overload represented by the curvilinear triangle *abd*. The area of this triangle is greater and is to be deducted from a more reduced economic advantage; thus it would be desirable

to aim at not too great a rate. This conclusion is modified, however, if we take into consideration the equipment costs, not as a whole, but separately. What matters, in effect, is that the considerable extra costs due to excessive multiplication of the number of wells should be avoided. On the contrary, right from the start, a connecting pipe of suitable diameter should be adopted, for the mean cost of a second pipe per m^3/h is clearly greater than the marginal cost incurred through adopting a larger diameter for the first pipe. Since supply of gas is indispensable in the first injection tests, it will be an advantage to adopt a fairly large diameter for this pipe. Conversely, there is no advantage at this initial stage in providing too many operating wells, because it will be very easy to increase the number of wells at a later date and ensure their completion so as to achieve the optimum conditions.

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A correction to this study is necessary. We have considered normal withdrawal needs only, without allowing for cases where withdrawal becomes necessary as a result of incidents at a production unit. It is clear, however, that the method used is applicable in such cases, provided that we specify the necessary emergency rate and the anticipated repair time or, in brief, the characteristics, of the accidental element that may occur and that we wish to allow for.

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Optimal Reliability of Parallel Multicomponent Systems

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In this paper the reliability of parallel multi-component systems will be considered.

The number and kind of spare components will be determined for maximum improvement in reliability under the assumption that limited funds are available for the purpose.

DEFINITION OF THE PROBLEM

A PROBLEM arising with electronic equipment is that of the reliability of the installation. An installation is to be considered as a system of a number of elements connected in series. Reliability is defined as the probability of successful operation.

This reliability can be improved by having in stock spare parts to replace the elements that have failed. In many cases there are not only one but several identical installations present. The spare parts can be used for any of the installations.

In this paper it is shown in which way the number of spare parts should be chosen if only a limited amount of capital is available for this purpose. The number of components is determined in such a way that the probability of successful operation of the installation is as great as possible, subject to given restrictions.

DETERMINING THE RELIABILITY OF THE SYSTEM

Suppose that each of the installations is composed of r elements† connected in series—represented by Figure 1—as squares connected by strokes. In Figure 2 is given a scheme of m parallel installations consisting of r phases and n_i spare components per phase.

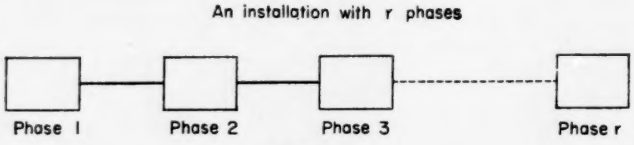

FIG. 1.

First of all, the most simple situation will be considered. This is the case when only one installation is present. Let ϕ represent the reliability of the

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† The list of symbols used are to be found at the end of this paper.

whole system and P_i that of phase i; then ϕ can be expressed by

$$\phi = \prod_{i=1}^{r} P_i,$$

provided that failures occur independently. Now the reliability P_i of each phase is dependent on the reliability p_i of each component and of the number of spare parts n_i.

If there is only one spare part, the phase that we are considering is not functioning properly when neither the component itself nor the spare part operate according to specifications, therefore

$$P_i = 1-(1-p_i)^2 \qquad (m = 1, n_i = 1).$$

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If there are n_i spare parts it is clear that

$$P_i = 1-(1-p_i)^{n_i+1}.$$

Now suppose there are two installations and one spare part, which can be used for both installations.

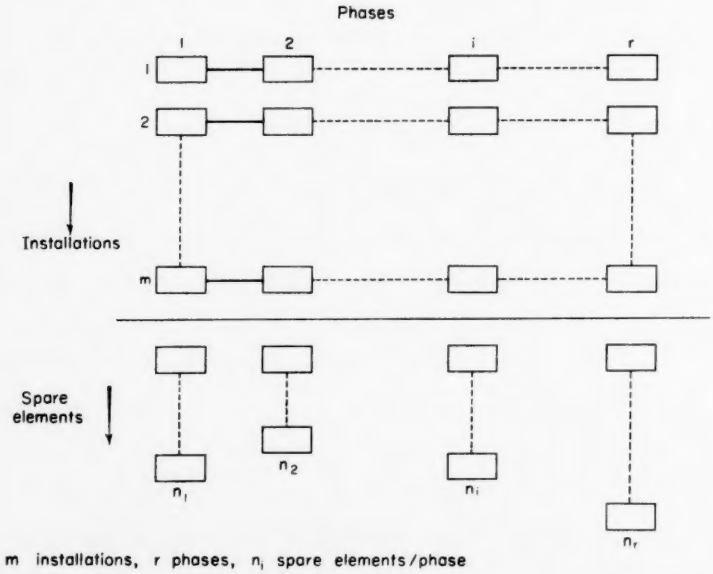

FIG. 2.

The probability that phase i of one of the installations does not operate properly consists of the following possibilities:

(1) The element of the installation we are considering and the spare part are out of order. The probability of this situation is $(1-p_i)^2$.

(2) The element itself is out of order, the spare part is in order but has been used because the other installation has failed. The probability of this situation is $(1-p_i)p_i(1-p_i)$.

However, the availability of the spare part also depends on the moment at which either system broke down. The system that fails first can make use of the spare part, the other cannot. Therefore the probability that this situation leads to stagnation of the installation is equal to

$$\tfrac{1}{2}(1-p_i)p_i(1-p_i).$$

So the probability that phase i fails to operate is equal to the sum of the probabilities under 1 and 2

$$P_i = 1-(1-p_i)^2-\tfrac{1}{2}p_i(1-p_i)^2.$$

In general, there are m parallel installations—r phases and n_i spare parts per phase. In the same way, P_i can be determined as a function of m and n_i (see Table 1). It is, however, easier to do this as follows: let h_i represent the number of defective elements and further $m+n_i = k_i$, and $Q_i = 1-P_i$.

TABLE 1. CHANCE OF STAGNATION (Q_i) FOR SOME VALUES OF m AND n

	$n=1$	$n=2$	$n=3\rightarrow$
$m=1$	$(1-p)^2$	$(1-p)^3$	$(1-p)^4$
$m=2$	$\frac{1}{2}(1-p)^2(2+p)$	$\frac{1}{2}(1-p)^3(2+2p)$	$\frac{1}{2}(1-p)^4(2+3p)$
$m=3$	$\frac{1}{3}(1-p)^2(3+2p+p^2)$	$\frac{1}{3}(1-p)^3(3+4p+3p^2)$	$\frac{1}{3}(1-p)^4(3+6p+6p^2)$
$m=4$	$\frac{1}{4}(1-p)^2(4+3p+2p^2+p^3)$	$\frac{1}{4}(1-p)^3(4+6p+6p^2+4p^3)$	$\frac{1}{4}(1-p)^4(4+9p+12p^2+10p^3)$
$\downarrow$			

The probability that h_i of the k_i elements are out of order is given by

$$R_{(h_i)} = \frac{k_i!}{h_i!(k_i-h_i)!}p_i^{k_i-h_i}(1-p_i)^{h_i}. \tag{1}$$

If $h_i \leqslant n_i$, the i^{th} phase of every installation works. If, on the other hand, $h_i = n_i+1$, then one of the m installations will be out of order. The probability that the i^{th} phase of the considered installation does not operate can be expressed by

$$Q_{i,1} = \frac{1}{m}R_{(h_i)} = \frac{1}{m}R_{(n_i+1)} \tag{2}$$

because this probability is equal for all the installations. The probability that

two installations are out of order is called $R_{(n_i+2)}$, the probability that the installation considered will be out of order is given by

$$Q_{i,2} = \frac{2}{m} R_{(n_i+2)}. \tag{3}$$

Generally speaking, the probability that the installation in question does not operate when n_i+s of the k_i elements are out of order, is given by

$$Q_{i,s} = \frac{s}{m} R_{(n_i+s)} \; (s \geqslant 0),$$

$$Q_{i,s} = \frac{s}{m} \frac{k_i!}{(n_i+s)!\,(k_i-n_i-s)!} p_i^{k_i-n_i-s} (1-p_i)^{n_i+s}, \tag{4}$$

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or

$$Q_{i,s} = \frac{s}{m} \frac{(m+n_i)!}{(n_i+s)!\,(m-s)!} p_i^{m-s} (1-p_i)^{n_i+s}. \tag{5}$$

The total chance that the i^{th} element of a certain installation of the system does not operate is equal to the sum of the probabilities of stagnations with $s = 1, 2, \ldots, m$

$$Q_i = \sum_{i=1}^{m} \frac{s}{m} \frac{(m+n_i)!}{(n_i+s)!\,(m-s)!} p_i^{m-s} (1-p_i)^{n_i+s}. \tag{6}$$

DISCUSSION

If one installation does not operate as a result of the failing of a certain phase, the elements of the phases which are working successfully could be used for other installations. However, this possibility has not been considered. The reliability has been defined as the probability of successful operation. The foregoing will do if the repair time is small in respect of the working time, which will be the case in most practical situations.

FINDING THE OPTIMUM OF RELIABILITY

In the introduction we have already stated that the reliability of the system should be maximized under the constraint of invested capital in spare parts (C). If the costs of one spare part of the i^{th} phase is equal to c_i, then

$$\sum_{i=1}^{r} c_i n_i \leqslant C \tag{7}$$

where $n_1, n_2, \ldots, n_r$ should be chosen such that

$$\phi = \prod_{i=1}^{r} P_i$$

is a maximum.

This value is called ϕ^* and is greater than every other ϕ. In consequence $\log\phi^*$ is greater than every other $\log\phi$

$$\log\phi^* = \max \log \prod_{i=1}^{r} P_i = \max \sum_{i=1}^{r} \log P_i. \tag{8}$$

As a result of the introduction of $\log\phi$ instead of ϕ it is possible to write the product as a sum.

The number of spare elements of the i^{th} phase is determined by

$$n_i = 0x_{i0} + 1.x_{i1} + 2.x_{i2} + \ldots + j.x_{ij} + \ldots \quad (j = 0, 1, 2, \ldots, \infty) \tag{9}$$

where
$$x_{ij} = x_{ij}^2, \tag{10}$$
and further,
$$\sum_{j=0}^{\infty} x_{ij} = 1, \tag{11}$$

which means that for every value of i one of the values of x_{ij} equals 1, all the others being 0. Now

$$c_i n_i = 0c_i x_{i0} + 1.c_i.x_{i1} + 2c_i.x_{i2} + \ldots + jc_i x_{ij} + \ldots \quad (j = 0, 1, \ldots, \infty). \tag{12}$$

If
$$jc_i = c_{ij}, \tag{13}$$
it follows that
$$c_i n_i = \sum_{j=0}^{\infty} x_{ij} c_{ij}, \tag{14}$$
so that
$$\sum_{i=1}^{r} c_i n_i = \sum_{i=1}^{r} \sum_{j=0}^{\infty} x_{ij} c_{ij} \leqslant C. \tag{15}$$

x_{ij} can also be introduced in the function which has to be optimized, e.g.

$$\log P_i = \sum_{j=0}^{\infty} x_{ij} \log P_{ij}, \tag{16}$$

where P_{ij} equals P_i with j spare parts in the i^{th} phase. It follows that

$$\sum_{i=1}^{r} \log P_i = \sum_{i=1}^{r} . \sum_{j=0}^{\infty} x_{ij} \log P_{ij}. \tag{17}$$

The problem defined by equations (15) and (17) is known as the knapsack problem†. We define

$$\Delta \log P_{i,j} = -\log P_{i,j-1} + \log P_{i,j}. \tag{18}$$

† A description of the problem can be found in: "The Journal of the Operations Research Society of America" under the title "Discrete-variable extremum problems" by G. B. Dantzig (Operations Research, April 1957).

For $\log P_{ij}$ can be written

$$\log P_{i,j} = \log P_{i,0} + (-\log P_{i,0} + \log P_{i,1}) + \ldots + (-\log P_{i,l-1} + \log P_{i,l}) + \ldots + (-\log P_{i,j-1} + \log P_{i,j}), \quad (19)$$

or

$$\log P_{i,j} = \log P_{i,0} + \sum_{l=1}^{j} \Delta \log P_{i,l}. \quad (20)$$

For formula (17),

$$\log \phi = \sum_{i=1}^{r} \log P_i = \sum_{i=1}^{r} \cdot \sum_{j=0}^{\infty} x_{ij} \log P_{ij} \quad (17)$$

can now be written

$$\log \phi \sum_{i=1}^{r} \cdot \sum_{j=0}^{\infty} x_{ij} (\log P_{i,0} + \sum_{l=1}^{j} \Delta \log p_{i,l}). \quad (21)$$

Log $P_{i,j}$, belonging to a given point, can be read off the graph 3 by adding the values of $\Delta \log P_{i,j}$ of the passed points and $\log P_{i,0}$.

These Δ's and their sum have been collected in the table (see Table 2). It should be noticed that the values of $\log P_{i,0}$ are situated on the horizontal axis because with $j = 0$ (no spare parts) the value of $c_{ij} = jc_i = 0$. Through this the constraint (11), viz.

$$\sum_{j=0}^{\infty} x_{ij} = 1$$

is fulfilled because always one point of every phase is chosen.

The variables in this kind of problem, however, are discrete, so that this method does not necessarily give the optimal answer. If one reads for C an amount *approximately* C and moreover is the proportion c_i/C small, then in practical situations the optimal solution will be found by rounding off.

Example

This method will be illustrated by an example of a system with 5 parallel installations each consisting of 3 phases. The prices of the spare components of the 3 phases are

$$c_1 = Hfl\ 100{\cdot}- \quad c_2 = Hfl\ 175{\cdot}- \quad c_3 = Hfl\ 500{\cdot}-.$$

The reliability of each component per phase is

$$p_1 = 0{\cdot}95, \quad p_2 = 0{\cdot}80, \quad p_3 = 0{\cdot}90.$$

The solution is found by means of a graph and Table 2. The value of $\Delta \log P_{ij}$ set out along the horizontal axis, along the vertical axis the value of c_i (Figure 3).

The value of $c_2 = 175$ (Table 2) can be found by $m = 5$ and $p = 0{\cdot}80$

$$n = 1 \rightarrow \Delta \log P = 548{\cdot}10^{-4}, \quad n = 2 \rightarrow \Delta \log P = 256{\cdot}10^{-4}, \quad \text{etc.}$$

TABLE 2. RELIABILITY PER PHASE (P_i) AS A FUNCTION OF m, n AND p

	$m = 1$			$m = 1$		
	$p = 0{\cdot}99$			$p = 0{\cdot}95$		
n	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$
0	9,900	44		9,500	223	
1	9,999		43	9,975	11	212
2	10,000			9,999	1	10
3				10,000		1
4						
5						

	$m = 1$			$m = 1$		
	$p = 0{\cdot}90$			$p = 0{\cdot}80$		
n	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$
0	9,000	458		8,000	969	
1	9,900	44	414	9,600	177	792
2	9,990	4	39	9,920	35	142
3	9,999		4	9,984	7	28
4	10,000			9,997	1	6
5				9,999		1

	$m = 2$			$m = 2$		
	$p = 0{\cdot}99$			$p = 0{\cdot}95$		
n	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$
0	9,900	44		9,500	223	
1	9,999	1	43	9,963	16	207
2	10,000		1	9,998	1	15
3				10,000		1
4						
5						
6						

	$m = 2$			$m = 2$		
	$p = 0{\cdot}90$			$p = 0{\cdot}80$		
n	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$
0	9,000	458		8,000	969	
1	9,855	63	394	9,440	250	719
2	9,981	8	55	9,856	63	187
3	9,998	1	7	9,965	15	48
4	10,000		1	9,992	4	12
5				9,998	1	3
6				10,000		1

TABLE 2—*Continued*

	$m = 5$			$m = 5$		
	$p = 0{\cdot}99$			$p = 0{\cdot}95$		
n	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$
0	9,900	44		9,500	223	
1	9,997	1	42	9,881	52	171
2	10,000		1	9,978	10	42
3				9,997	1	8
4				10,000		1
5						
6						
7						
8						

	$m = 5$			$m = 5$		
	$p = 0{\cdot}90$			$p = 0{\cdot}80$		
n	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$	$P \times 10^4$	$-\log P \times 10^4$	$\Delta \log P \times 10^4$
0	9,000	458		8,000	969	
1	9,737	116	342	9,076	421	548
2	9,943	25	91	9,627	165	256
3	9,989	5	20	9,864	59	106
4	9,998	1	4	9,954	20	39
5	10,000		1	9,985	6	14
6				9,996	2	4
7				9,999	1	1
8				10,000		

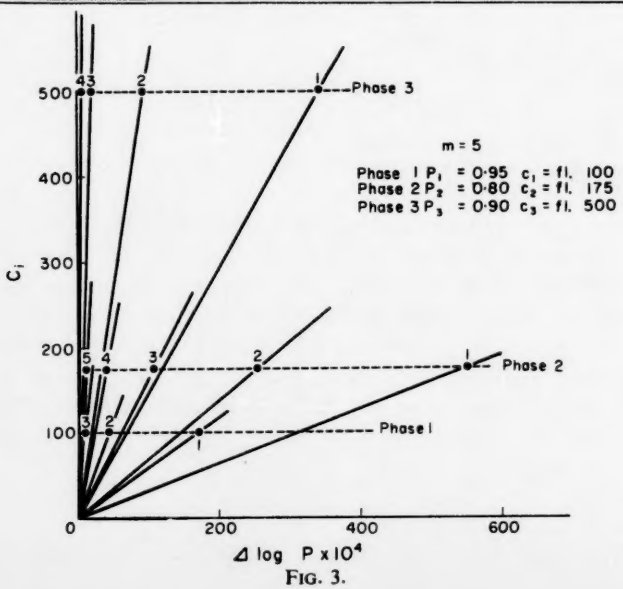

FIG. 3.

The point in the graph for one spare component in the second phase is found at the point where $c_2 = 175$ and $(\log P_{2,1}) \cdot 10^4 = 548$, and the reliability in the second phase with one spare component is $P_{2,1} = 0{\cdot}9076$ (see Table 2).

In the same way the point for $n = 2$ is found where $c_2 = 175$ and $(\Delta \log P_{2,2}) \cdot 10^4 = 256$, where $P_{2,2}$ has become equal to 0·9627. Proceeding in this way, the whole graph can be constructed. The solution for several values of C can be obtained by rotating a straight line on the origin.

If the line passes a point in the graph the number of spare components indicated by the point should be purchased. The total cost is raised by the amount c_i.

By applying the proposed method in this example the optimal choice of spare components will be as shown in Table 3.

TABLE 3. TO BE READ IN CONJUNCTION WITH FIGURE 4

Number of spare components	phase 1 $c_1 = 100$ $p_1 = 0{\cdot}95$	phase 2 $c_2 = 175$ $p_2 = 0{\cdot}80$	phase 3 $c_3 = 500$ $p_3 = 0{\cdot}90$	$\log \phi$	ϕ	C total
0	−0·0223	−0·0969	−0·0458	0·8350–1	0·6769	*Hfl*· 000·–
1		−0·0548		0·8898–1	0·7759	*Hfl*· 175·–
2	−0·0171			0·9069–1	0·8071	*Hfl*· 275·–
3		−0·0256		0·9325–1	0·8561	*Hfl*· 450·–
4			−0·0342	0·9667–1	0·9262	*Hfl*· 950·–
5		−0·0106		0·9773–1	0·9491	*Hfl*· 1125·–
6	−0·0042			0·9815–1	0·9583	*Hfl*· 1225·–
7		−0·0039		0·9854–1	0·9669	*Hfl*· 1400·–
8			−0·0091	0·9945–1	0·9874	*Hfl*· 1900·–
9	−0·0008			0·9953–1	0·9892	*Hfl*· 2000·–
10		−0·0014		0·9967–1	0·9924	*Hfl*· 2175·–
11			−0·0020	0·9987–1	0·9970	*Hfl*· 2675·–
12		−0·0004		0·9991–1	0·9979	*Hfl*· 2850·–
13	−0·0001			0·9992–1	0·9982	*Hfl*· 2950·–
14			−0·0004	0·9996–1	0·9991	*Hfl*· 3450·–
15		−0·0001		0·9997–1	0·9993	*Hfl*· 3625·–
16			−0·0001	0·9998–1	0·9996	*Hfl*· 4125·–

From this it appears that first of all one spare component of phase number 2 has to be purchased, component 1 follows next, then component 2 and so on. The reliability and the total costs can be read every time in the columns (Table 3) and the angle is determined by the quotient $c_i/\Delta \log P_i$ (Figure 4). With small values of $\Delta \log P_i$ there are difficulties as a result of inaccuracies of the drawing, therefore, it is easier to use the quotients of the points only. By arranging these quotients in order of value (the smallest first) a method is obtained to determine the spare components to be purchased (Table 4).

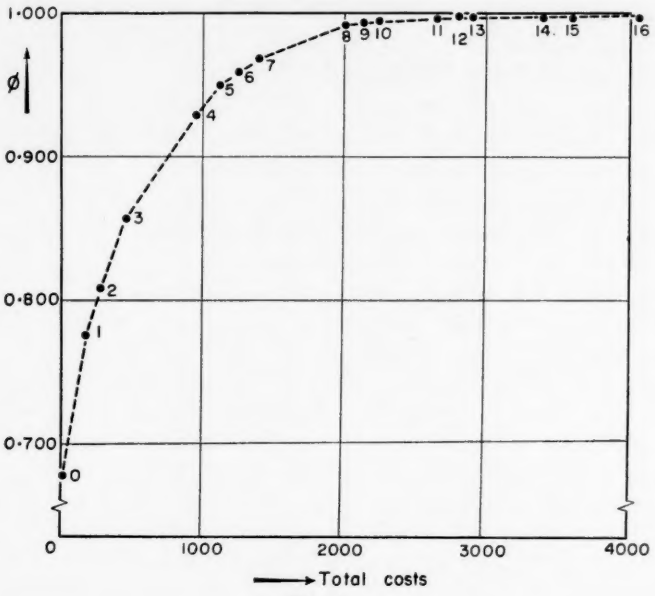

FIG. 4.

TABLE 4. ARITHMETICAL METHOD TO MAXIMIZE THE RELIABILITY

Number of spare components	c_1	$\log P_i$	$= \frac{c_i}{d \log P_i}$
1	175	0·0548	3·193
2	100	0·0171	5·848
3	175	0·0256	6·836
4	500	0·0342	14·620
5	175	0·0106	16·509
6	100	0·0042	23·810
7	175	0·0039	44·872
8	500	0·0091	54·945
9	100	0·0008	125·000
10	175	0·0014	125·000
11	500	0·0020	250·000
12	175	0·0004	437·500
13	100	0·0001	1·000·000
14	500	0·0004	1·250·000
15	175	0·0001	1·750·000
16	500	0·0001	5·000·000

LIST OF SYMBOLS

The following symbols are used in this paper:

- ϕ the reliability of the whole system,
- P_i the reliability of the i^{th} phase,
- $P_{i,j}$ the reliability of the i^{th} phase if this phase has j spare components,
- p_i the reliability of one component in the i^{th} phase,
- m the number of parallel installations,
- n_i the number of spare components in the i^{th} phase,
- r the number of phases of the system,
- h_i the number of defective components in the i^{th} phase,
- k_i the total number of components in the i^{th} phase—this is the sum of working and spare components ($= m+n_i$),
- $Q_{i,s}$ the probability that the i^{th} phase of an installation does not operate if s of the components are out of order,
- C total funds available to invest in spare components,
- c_i the price of one spare component in the i^{th} phase,
- x_{ij} indicates if a number of j components in the i^{th} phase is chosen or not, (= 0 or 1)
- c_{ij} the price of j spare components in the i^{th} phase,
- ΔP_{ij} the increase of the reliability if j spare components are chosen instead of $(j-1)$.

A Multiple Reorder Point Inventory Policy

ERNEST KOENIGSBERG

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This paper discusses the development and application of a multiple reorder inventory policy which can be stated as follows: reorder an optimal lot size Q when inventory (stock on hand) falls to $R, R-Q, R-2Q, \ldots, R-NQ$; where R is the reorder level. If demands cause the inventory to fall below two reorder levels, say a jump from $R+\epsilon$ to $R-2Q+\epsilon'$ where ϵ and $\epsilon' < Q$, an order for $2Q$ is placed. The policy is a form of (S, q) policy where the maximum stock level $S = R+Q$. The system is of particular value in cases where the coefficient of variation of lead time demand $\mu_l(\mu_l = \sigma_l/\lambda_l)$ is large (say $>0{\cdot}5$) and continuous inventory records are maintained. Tables, charts and nomographs to simplify clerical tasks can be obtained quite readily. In this formulation R and Q are not independent factors as in the usual Wilson formulation, but are obtained by minimizing a single cost functional subject to the constraint of a specified risk of out-of-stock condition or a specified level of service (Galliher and Simmond, 1957), (Morse *et al.*, 1959). The particular application concerns the raw material inventories of a manufacturer of metal pressings who is required to offer "immediate service". The demand distribution during the lead time closely approximates the exponential distribution, and lead times are constant for each raw material. The application of the multiple reorder policy results in a 30 to 35 per cent reduction in inventory for a 95 per cent service level. Measures of sensitivity and response are obtained, and the mean number of shortages is expressed in closed form. The policy is compared with the Wilson policy and shown to be more "effective" in that it results in lower inventories and a smaller number of orders for the case considered.

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MOST studies of inventory and reorder policies are developed on the basis of a single reorder point. In the Wilson system, the optimum lot quantity is selected by minimizing a cost function and the safety stock level is then selected as being some multiple of the standard deviation of demands during the lead time. In the (S, s) model, both the maximum stock level (S) and the single reorder point (s) are selected by minimizing a cost function. In this latter policy, multiple reorder points and reorder quantities are possible if the cost function has a number of relative minima.

The policy recommended in this paper has a very simple form; reorder an optimal lot size Q when the inventory falls to $R, R-Q, R-2Q, R-NQ$ (if there is a double jump, say from $R+\epsilon$ to $R-2Q+\epsilon'$, where ϵ and ϵ' are less than Q, then we order $2Q$). The maximum stock level (S in the (S, s) policy) is $R+Q$. This system has a number of advantages, not the least of which is the simplicity of record keeping. It results in lower inventories than are obtained in the Wilson system, and generally requires that fewer orders be placed. The system has particular advantages when the variation in demand during the lead time is large, of the order of the mean lead time demand or greater, as occurred in the example described here. The model was originally developed (Galliher and Simmond, 1957) for problems in military inventories and has since been

extended in a more general manner (Morse *et al.*, 1959). The case discussed here is a commercial application.

"The X Company"† makes sheet-metal pressings from a number of metals of different composition, each in a range of sheet sizes and thicknesses. The company must operate on an immediate service basis; i.e. orders are delivered to customers within two or three weeks of the receipt of the orders, while the lead time for raw materials generally exceeds four weeks. The company is a "job shop" in that each customer sets specifications which can and do vary, and therefore no finished goods inventories can be maintained. To meet the demands of customers, the company carries a large stock of sheets of the standard sizes. Since this stock represents a considerable inventory investment, the company is interested in reducing it if such reductions can be obtained without prejudicing customer service.

At the time the study began, company policy called for a three months' (anticipated) supply of all standard sizes and finishes to be held. Inventory was "managed" by a very efficient purchasing agent who endeavoured to keep stock levels within the three-month usage range and to avoid shortages. Two inventory records were kept, one of the inventory on hand, the other of inventory available for new orders (on-hand inventory minus stocks already committed to orders) as well as a list of unfilled orders. The purchasing agent was in contact with the sales department in order to develop forecasts of raw material requirements, and in particular to obtain advance information on orders requiring sizes and finishes not carried in the normal inventory.

A study of the demands on inventory developed the following information. Orders for any particular sheet were quite random, with many orders for small quantities (say 1000 or 2000 sheets) and a few large orders (as high as 150,000 sheets). Analysis showed that demands during the delivery lead time followed an exponential distribution quite closely for most standard items. Figure 1 is a plot of the standard deviation of the demand during the lead time (σ_l) as a function of the mean demand during the lead time (λ_l). The "best fit" line has a slope of one; i.e. $\sigma_l = \lambda_l$. For the exponential distribution, $\sigma_l \equiv \lambda_l$. While this does not guarantee an exponential distribution, further confirmation was obtained by analysing data for the past three years for more than 20 standard inventory items. The demand curve could be approximated over most of the range by an exponential distribution. There were two sheet sizes which deviated significantly from the exponential distribution. Both were newly developed sheets which were being used frequently in small experimental runs and had not yet received general acceptance by customers.

† The company is fictitious. The actual study was carried out for a company in a different industry, thus the cost factors and mode of operation do not fit the pattern of the metal fabricating industry. The problem, however, is a very real one and has factors in common with metal fabricating and other industries.

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Standard lead times for each item were obtained from the purchasing department. A study of inventory records for the past three years showed that lead times were extremely stable. The few exceptions to this constancy were checked and in each case the difference could be attributed to request for speedier delivery (or delayed delivery) made by the X Company. As a result of this analysis, lead times were considered as fixed for each sheet (they varied from three weeks to ten weeks for the individual items in inventory).

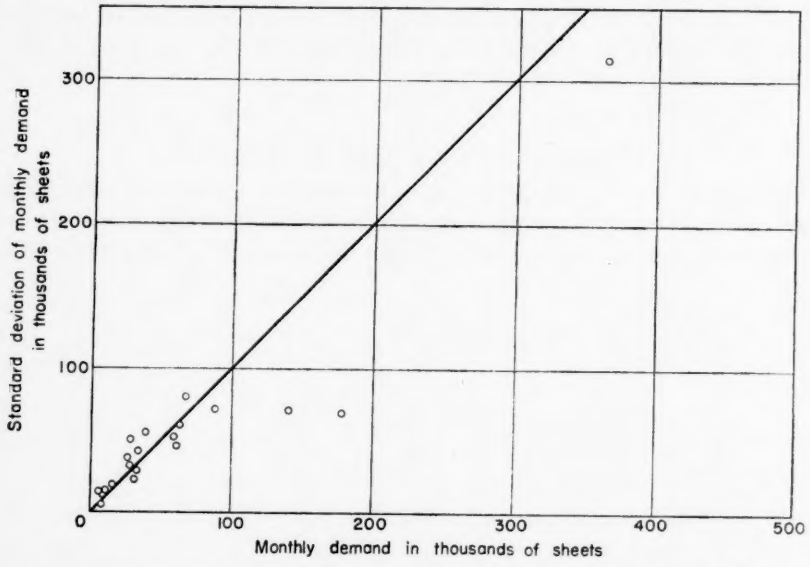

FIG. 1. *Standard deviation of demand during lead time,* σ_l *(in units of* 1000).

Forecasts of demands were made quarterly by the purchasing agent in conjunction with the sales department. The forecasts were quite reliable for the total production requirement but were not extremely accurate for the individual items. Forecasts were revised, when necessary, by the purchasing department after discussions with the sales department and a review of past usage rates.

Then, given an exponential distribution of demand during the lead time, a constant lead time, and the desire and need to maintain a high level of service, we seek an inventory management policy which can achieve these goals at a minimum cost. The large standard deviation of lead time demand makes the usual Wilson formulation somewhat inadequate, particularly in view of the fact that the number of units in a single order may be larger than Q^*, the Wilson lot size and also than σ_l, the variation of demand during the lead time. Within the cost range, Q^* is frequently less than σ_l, and the single reorder

policy would require separate records of stocks on order for proper maintenance of inventory control. The multiple reorder policy is well suited to the conditions of the problem.

One can determine a policy (1) based on a prescribed standard of service, i.e. the probability of being out of stock is set by company policy and one minimizes the cost of inventory operations, or (2) by assessing the cost of back orders (or being out of stock) and adding it to the cost function to be minimized. This latter method will determine the value of P, the probability of being out of stock. Since the cost of stock-outs is, at the very least, difficult to determine, the first alternative is used most frequently. This takes the desirable form of maximum service at least cost.

THE MULTIPLE REORDER POLICY

Figure 2 is a plot of a typical inventory history over a period of time. Stock levels fluctuate from a maximum value (indicated here by $R+Q$) to a minimum value, which may be negative. R is the reorder point and Q is the lot size, i.e.

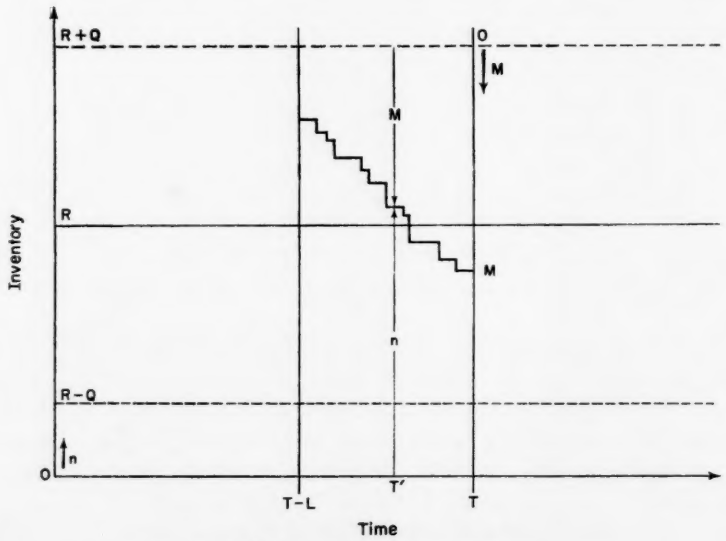

FIG. 2. *A plot of a typical day-to-day inventory.*

when the inventory levels fall to R, an order for Q units is placed. If there are no demands during the lead time, the stock level will attain its maximum when the order arrives. If the stock level n falls to $R-2Q$ at some time short of the lead time, a second order must be placed. The second order is necessary because when the first order arrives the stock level will still be less than the reorder point.

Following Galliher and Simonds (1957) we introduce a new variable M which we call the "reduced inventory" given by

$$M = R+Q-n = \text{maximum inventory} - \text{actual inventory}. \tag{1}$$

Now, if l is the lead time, any orders outstanding at time $T-l$ would have arrived by time T. Then we can consider the value of M at time T to be the value of M at time $T-l$, plus the outstanding orders minus the demand in the interval $(T-l, T)$. The inventory on hand plus on order, measured by the reduced inventory can only vary from zero to $Q-1$, for when we reach Q we reorder and jump back to zero. If we write $d(y,l)$ as the probability that there will be a demand for y units in a time interval l, we obtain

$$p(M) = \frac{1}{Q}\sum_{0}^{M} d(y,l) \quad (M<Q),$$

$$p(M) = \frac{1}{Q}\sum_{M-Q+1}^{M} d(y,l) \quad (M \geqslant Q), \tag{2}$$

where $p(M)$ is the probability that the reduced inventory has the value M. The mean value of the reduced inventory is

$$\bar{M} = \sum_{M=0}^{\infty} Mp(M) = \frac{Q-1}{2} + \lambda_l, \tag{3}$$

where λ_l is the mean demand during the lead time l (λ is mean demand rate). The variance of the reduced inventory is given by

$$\sigma_M^2 = \sigma_l^2 + \frac{Q^2-1}{12}, \tag{4}$$

where σ_l is the standard deviation of demand during the lead time.

Recognizing that shortages or stock-outs will occur, and setting

$$R+Q = \bar{M} + k\sigma_M,$$

the mean inventory is given by

$$\bar{n} = R + \frac{Q+1}{2} - \lambda_l + \sum_{R+Q}^{\infty} [M-(R+Q)]p(M). \tag{5}$$

The last term can be shown to be quite small under the assumption that the tail of the distribution function $p(M)$ is normal or exponential and $k > 1{\cdot}5$ or 2. Ignoring this term, we find

$$\bar{n} = R + \frac{Q+1}{2} - \lambda_l = k\sigma_M = k\sqrt{\left(\sigma_l^2 + \frac{Q^2-1}{12}\right)}. \tag{6}$$

Under the policy of setting a standard of service to be maintained, we can write the total variable operating cost as

$$\text{T.V.C.} = C\bar{n} + \frac{\lambda L}{Q} S = kC\sqrt{\left(\sigma_l^2 + \frac{Q^2}{12}\right)} + \frac{\lambda L}{Q} S, \tag{7}$$

where C is the cost per unit of holding inventory over time period L, S is the setup or ordering cost, $\lambda L/Q$ is the expected number of reorders, and k is a measure of the standard of service. Differentiating with respect to Q to obtain the minimum cost, we get

$$Q^3 = \frac{6}{k}\frac{2\lambda LS}{C}\sqrt{\left(\sigma_l^2 + \frac{Q^2}{12}\right)}. \tag{8}$$

We recognize that Q^*, given by

$$Q^* = \sqrt{\left(\frac{2\lambda LS}{C}\right)}, \tag{9}$$

is the Wilson lot size. Then we write

$$Q = \alpha Q^*, \tag{10}$$

where α is given by

$$\alpha^6 - \frac{3}{k^2}\alpha^2 = \left(\frac{6\sigma_l}{kQ^*}\right)^2. \tag{11}$$

Then the system is defined by Eqs. (6), (9) and (11).

APPLICATION OF THE POLICY

The policy can be applied by calculating λ_l and σ_l for each type of material or by making use of the fact that $\lambda_l = \sigma_l$ for the exponential distribution. Since most of the raw materials do in fact approximate to an exponential distribution of demand during the lead time, the latter course would be preferable for the ease of calculation (particularly as demand rates change) and for simplicity of operation. The two methods were compared and the small cost difference indicated that the simpler method could be used. It was suggested, however, that for those items for which the standard deviation was considerably less than the mean demand, the first method would result in considerable advantage in terms of reduced inventory.

Given λ, l, C, S and k, one can readily evaluate the decision variables R and Q from Eqs. (6), (9) and (11). Alternatively, one could use nomographs and charts for prescribed values of k and S. R and $\bar{n}$ are not strongly dependent on C (for a given k). This suggests the use of a simple graph (or a straight line equation) to determine R and $\bar{n}$ as functions of λ_l. Figure 3 is a plot of R as a function of λ_l. The hatched area shows the variation in R as C changes from \$1.20 to \$30.00 per thousand sheets per year. The circles indicate the values using the actual σ_l rather than $\sigma_l = \lambda_l$. The value of R changes by about

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7 per cent when C changes by a factor of 25. The raw material sheets which show significant deviation from the suggested linear relationship are those discussed previously, used largely for "experimental" pressings. It might be noted here that the slight dependence of R on the cost factors also serves to make the Wilson approach effective.

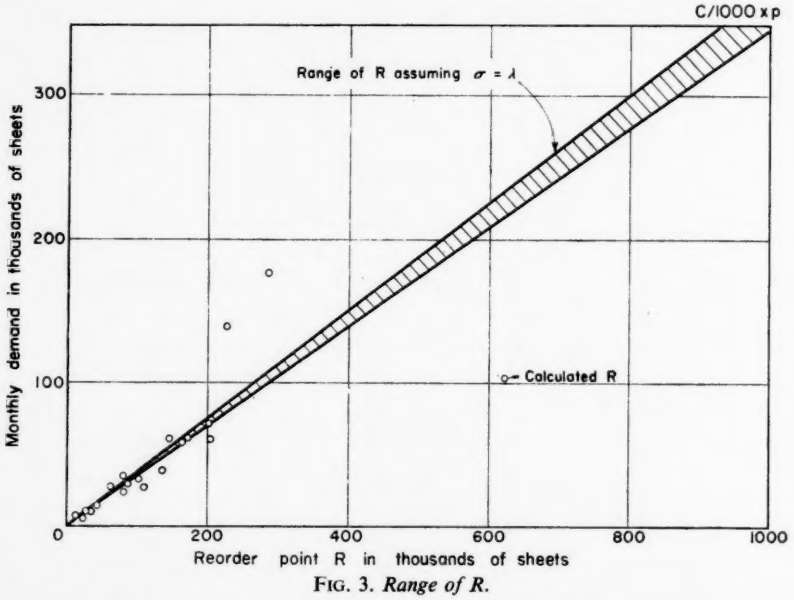

FIG. 3. *Range of R.*

As might be expected, Q, the lot size, is strongly dependent on the cost factors. Q can be determined from a graph of α (see Eq. (11)) as a function of $6\sigma_l/kQ^*$, and the calculation of Q^* by taking the square root. Alternatively, one could develop a nomogram of the relation between Q and C and λ_l (for given values of S and k). This was done for ease of calculation and the nomogram is shown in Figure 4. Where calibre of clerical staff is lower, one can develop tables of Q as a function of C and λ_l.

Having established the decision variables, R and Q, for each inventory item, inventory control is a relatively straightforward matter. The record card contains the decision variables R and Q (and $R-Q, R-2Q, \ldots, 1$). When items are allocated to an order, the amount of the allotment is deducted. If the resultant value of inventory available crosses one (or more) reorder points, an order for Q ($2Q$ or more) units is placed. When orders are received, the quantity is added to the available inventory. The clerk need not refer to outstanding orders or do more than one subtraction (or addition) per allotment (or order received). The net stock is recorded in an adjoining column for checks and

also to allow emergency changes and transfers. The amount available is used as a control to give some added protection. This increases stock levels by, at most, five days' supply, but ensures that satisfactory production schedules can

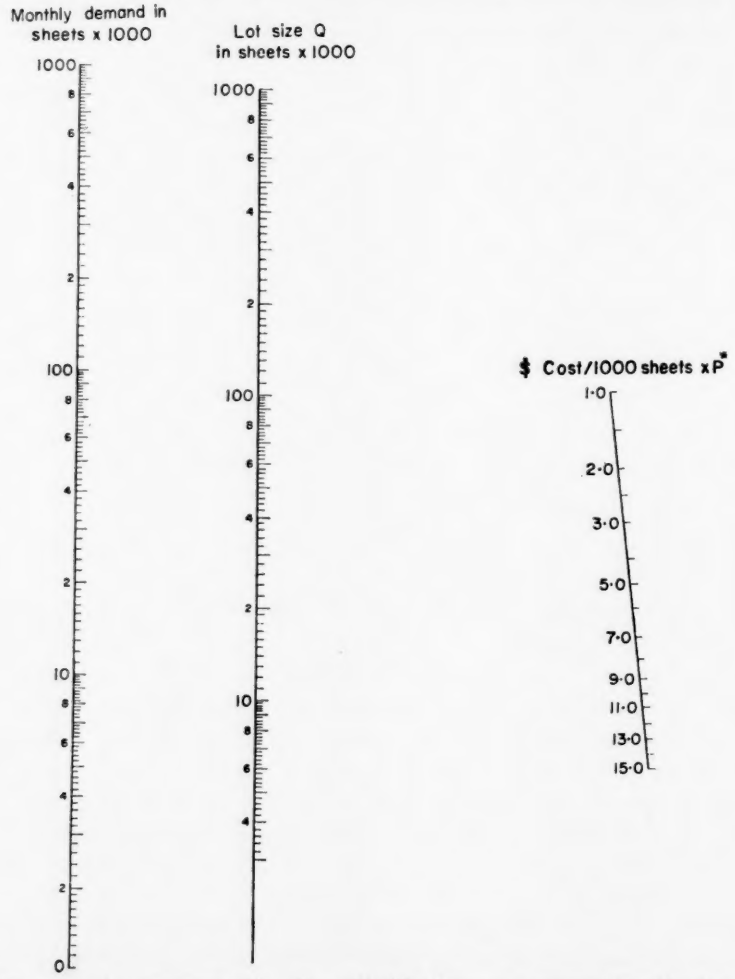

FIG. 4. *Lot size nomogram.* P^* = *Cost of holding inventory expressed as a fraction* (4 *per cent* = 0·04).

be run and that delivery promises are kept. Should demand rates, lead times or prices change, the new optimum decision variables can be calculated by clerical staff using the Q nomogram (or tables) and the R vs. λ_l chart.

To test the efficiency of this new policy, a simulation of 3 years of past history was run, using actual demand data and a rough forecast of demand.

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TABLE 1. COMPARISON OF REDUCTIONS FOR SEVERAL RAW MATERIALS

Raw material No.	Total demand in units of 1000 sheets	Years	Total shortages (Thousands of sheets)			Cost per 1,000 sheets ($)	Average inventory (Thousands of sheets)			Cost of inventory		
			Actual	Policy Service 95%	Policy Level 92%		Actual	Policy Service 95%	Policy Level 92%	Actual	Policy Service 95%	Policy Level 92%
12	2371	3	0	110	350	65·79	509	172	133	$33,500	$11,700	$8,750
7	1159	3	45	35	107	37·80	96	62	54	3620	2320	2040
1	818	3	0	40	88	58·70	170	52	41	9950	3050	2410
2	3196	3	0	320	550	69·89	200	111	86	13,950	7830	6000
11	11,753	3	565	210	460	35·35	707	736	611	25,000	26,000	21,600
8	412	3	—	81	102	113·25	64	42	34	7260	4760	3860
4	290	2	—	58	87	153·04	117	11	8	17,900	1690	1230
Total	19,999	20	610	854	1744		1863	1186	967	$111,180	$57,350	$45,890
			3·05%	4·27%	8·72%			−36·5%	−48·6%		−48·7%	−59·0%

The results (based on 12 sheet types for a total of 34 item years) were more than satisfactory. They indicated that, for $k = 2$, inventory could be reduced by more than 45 per cent with little change in the standard of service. (The existing records did not show all stock-outs which occurred in the 3-year period, nor did they indicate substitutions or special material purchases. Further, the simulation assumed that advance knowledge of customer orders was not available; in practice such forewarnings do occur.) The number of orders written changed by less than 5 per cent. Some sample results are listed in Table 1. The value of the inventory reduction was larger, in terms of dollars, than in the number of sheets held. The values in this table are based on the calculated σ_l rather than the assumption $\sigma_l = \lambda_l$.

SENSITIVITY AND RESPONSE OF THE POLICY

The cost of operation is not very sensitive to σ_l. We can write

$$\text{T.V.C.} = \sqrt{(2\lambda LSC)}\left(\frac{k\alpha^3}{6}+\frac{1}{2\alpha}\right). \tag{7'}$$

Letting $K = \sigma_l^2/\lambda_l$,

we can rewrite Eq. (8) as

$$\alpha^6 - \frac{3}{k^2}\alpha^2 = \left(\frac{6\sigma_l}{kQ^*}\right)^2 = \frac{18C'\sigma^2}{k^2 S\lambda_l} = \frac{18C'}{k^2 S}.K, \tag{12}$$

where $C'/l = C/L$. From this we get

$$\frac{dK}{K} = \frac{6(k^2\alpha^4-1)}{\alpha(k^2\alpha^4-3)}d\alpha. \tag{13}$$

Then

$$\frac{d(\text{T.V.C.})}{\text{T.V.C.}} = \frac{3(k^2\alpha^4-1)}{\alpha(k^2\alpha^4+3)}d\alpha. \tag{14}$$

Substituting for $d\alpha$ from Eq. (13), we obtain

$$\frac{d(\text{T.V.C.})}{\text{T.V.C.}} = \frac{1}{2}\frac{k^2\alpha^4-3}{k^2\alpha^4+3}.\frac{dK}{K}. \tag{15}$$

Then if α is close to 1 and $k = 2$, an error of 100 per cent in K (so that $dK/K = 1$) results in an error of 1/14 or 7 per cent in the total operating cost. The maximum error for large α is 50 per cent of the error in K. When $\alpha = 1{\cdot}5$ the error in (T.V.C.) for a 100 per cent error in dK/K is 37 per cent.

If K is known accurately, but there is an error in λ_l, we see from Eq. (7')

$$\frac{d(\text{T.V.C.})}{(\text{T.V.C.})} = \frac{1}{2}\frac{d(\lambda_l)}{\lambda_l}. \tag{16}$$

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Hence errors in λ_l are more serious than errors in σ_l, except when $\alpha \gg 1$. Then the limiting error in cost is also 50 per cent of dK/K. For α close to 1 the errors due to errors in σ_l are relatively small, but they increase rapidly to 50 per cent of the error in σ_l^2 and λ_l. This suggests that for large values of α (hence large values of $(6\sigma_l/kQ^*)$ one should use the actual value of σ_l rather than the estimated value, $\sigma_l = \lambda_l$. This implies more clerical effort. Alternatively, one could divide the inventory into groups according to the coefficient of variation σ_l/λ_l and have charts for each group.

The probability of a shortage and the mean number short can readily be calculated for this policy. From Eq. (2), we have the value of $p(M)$, the probability that the "reduced inventory" has the value M. Under the assumption that $d(y, l)$ is exponentially distributed with a mean value $\bar{y} = \lambda_l$, we find

$$p(M) = \frac{1}{\bar{y}} \cdot \frac{1}{Q} \int_{M-Q+1}^{M} e^{-y/\bar{y}}\, dy,$$

$$p(M) = \frac{e^{-M/\bar{y}}}{Q} [\exp(Q - 1/\bar{y}) - 1]. \tag{17}$$

Now a shortage will occur when M exceeds $(R+Q)$. The probability of a shortage is then the sum of $p(M)$ over all values of M greater than $R+Q$. Hence

$$P_{\text{out}} = \int_{R+Q}^{\infty} p(M)\, dM = \frac{\bar{y}}{Q} [\exp(Q - 1/\bar{y}) - 1] \exp[-(R+Q)/\bar{y}], \tag{18}$$

and the mean number of units short is

$$M_{\text{out}} = \int_{R+Q}^{\infty} Mp(M)\, dM = \bar{y} P_{\text{out}}. \tag{19}$$

Table 2 lists the estimated shortages for $k = 2$, as well as the shortages in the present system (where data were available), and the shortages obtained from the simulation model. The average probability of a shortage for $k = 2$ is 4·34 per cent for the calculation from Eq. (18), 3·43 per cent from the simulation of the policy, and 2·93 per cent from the actual data. The differences can be attributed to a number of factors. First, the simulation was carried out using recorded data rather than the approximation of an exponential distribution of demand. The tail of the exponential function is broader than the normal distribution and somewhat broader than the actual data. Thus one would expect Eq. (18) to yield a somewhat higher probability of shortage. The use of the exponential form for the demand distribution allows simple calculations of conservative standards which will be exceeded (bettered) in practice.

Part of the difference between the simulation and the actual (historical) performance can be attributed to the fact that not all shortages appeared in the records; i.e. when an order would result in a shortage, substitutions were sometimes made for a number of items (such as using a more expensive sheet

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which still satisfied the customers' requirements) and, at times, these orders were not listed until the material arrived in inventory. While such actions were not endorsed by the company, they did occur. It would be quite difficult, if not impossible, to reconstruct the actual events for a more precise comparison. The difference in the probability of stock-out for the simulation and the actual data is reasonably small, and types of errors made would tend to reduce this difference further. In view of the examination above, Eqs. (18) and (19) appear to be reasonably good measures of expected shortages and give adequate value for management's evaluation of service standards.

TABLE 2. EXPECTED PER CENT SHORTAGE

Raw material No.	Estimated shortage (%)	Actual shortage (%)	"Policy" shortage (%)
1	3·56	—	4·90
2	4·38	—	10·0
3	4·55	—	—
4	4·62	—	20·0
5	3·50	—	—
6	4·26	—	—
7	4·80	3·88	3·02
8	4·15	—	20·0
9	5·06	—	—
10	4·01	—	—
11	5·26	4·81	1·79
12	3·94	—	4·70
Average	4·34	2·93	4·27

The system, while not always violently sensitive to the forecast (λ_l) is, however, sensitive enough to give rewards for accurate forecasts. Similarly, any steps which can be taken to reduce the standard deviation of demand σ_l will be equally rewarding. A reduction of σ_l from λ_l to $\lambda_l/2$ will reduce the average inventory by about 50 per cent (at $k = 2$) with the same standard of service. In some cases advance notices of customer orders will have the same effect as a reduced σ_l.

The study was carried out for $k = 2$ (equivalent to 2 protection in the Wilson model). Calculations were also made for $k = 1{\cdot}5$ for comparative purposes. Reducing k, of course, reduces the inventory level at the expense of greater shortages. The probability of stock-outs increased to over 8 per cent and the average inventory reduction increased from 36 per cent (for $k = 2$) to 48 per cent. In dollar value the inventory was reduced by 59 per cent. A service level of 92 per cent (8 per cent shortages) was felt intolerable by management, the 4 per cent level was close enough to existing practice to warrant the use of $k = 2$.

The advantages of this multiple reorder policy over the usual Wilson formulation are shown in Tables 3 and 4. The policy is compared with the Wilson

TABLE 3. COMPARISON OF Q, R, $\bar{n}$ FOR THE USUAL WILSON FORMULA AND THE PROPOSED MODEL

	Wilson formula			Model		
No.	Q (Thousands of sheets)	R	$\bar{n}$	Q (Thousands of sheets)	R	$\bar{n}$
1	29	66	78	42	87	70
2	54	213	305	105	390	280
3	29	138	149	54	180	140
4	9	21	20	12	18	17
5	25	165	91	42	110	84
6	43	216	198	77	235	182
7	36	87	84	49	77	70
8	15	54	39	22	35	40
9	42	207	377	96	485	370
10	20	96	70	32	80	62
11	124	939	782	250	950	745
12	39	240	154	67	185	145
13	27	126	82	42	90	75
14	25	114	65	46	67	58
15	55	196	164	83	178	142
16	27	150	70	40	75	60
17	16	42	28	19	25	25
18	17	39	29	18	20	20
19	46	156	143	71	160	125
20	15	39	24	17	20	20
Total			2952			2730
Per cent increase			8·13			—

TABLE 4. NUMBER OF ORDERS PLACED

Raw material No.	Actual	Theory	Wilson formula
1	13	17	27
2	33	52	63
3	26	15	56
4	14	22	21
7	29	38	32
8	14	15	25
11	81	54	97
12	34	34	61
Total	244	247	382
Per cent increase		1·23	56·50

policy in terms of average inventory level and the number of orders for a Wilson policy with a safety stock of 2. The Wilson policy requires an inventory about 8 per cent larger and results in 50 per cent more orders being placed. Both effects can be predicted from the basic equations which are :

$$\text{Wilson, } \bar{n} = \frac{Q^*}{2} + 2\sigma_l = 2\sigma_l\left(1 + \frac{Q^*}{4\sigma_l}\right);$$

$$\text{Multiple reorder, } \bar{n} = 2\sqrt{\left(\sigma_l^2 + \frac{Q^2}{12}\right)} \simeq 2\sigma_l\left[1 + \frac{1}{24}\left(\frac{\alpha Q^*}{\sigma_l}\right)^2\right].$$

Since σ_l is generally greater than Q^*, the average inventory for the Wilson policy will be larger. Now $Q = \alpha Q^*$, where Q^* is the Wilson lot size and $\alpha(1 \leqslant \alpha \leqslant 3)$ is given by Eq. (11). Since the Wilson lot size is smaller, there will be more orders placed, on the average α times as many, using the Wilson formula. The Wilson formula does not necessarily provide better service to compensate for the high inventory and greater operating costs.

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CONCLUSIONS

The multiple reorder policy has many advantages over the Wilson policy for this application. The simplicity of operation suggests that it is a better operational policy than (S, s) policy as usually applied. In principle, it provides decision rules for minimum cost service for a specified out-of-stock risk or level of service.

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Optimum Lot-size Policy if Tools Break Down Frequently

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A well-known formula gives the optimum length of a production series by balancing change-over costs against inventory costs. This formula can no longer be considered as optimum in the case of a high breakdown rate of the tools. We can often reduce our costs in such situations by ending the production series when a breakdown occurs. This introduces a variability in the lot-sizes, which in turn causes higher stock levels.

A production policy was developed, based on two limits, Q_1 and Q_2. Production is stopped either at the first break-down after Q_1 units have been produced, or at Q_2. Optimum values of Q_1 and Q_2 were derived and can be read off from a set of graphs.

One assumption of this theory is that tools are always available, which theoretically requires an infinite stock of these tools.

A Monte Carlo programme was set up to study the situation, in which the number of tools was limited. The number of tools, and the reorder-level for the products, required for a desired probability of stock-out, can be derived by the programme.

The theoretical values of Q_1 and Q_2 could be considered optimum for practical purposes in the cases under consideration.

1. INTRODUCTION

In a production shop a large number of products is made in series with machines of one type. Changeable tools determine the output of the machines. The tools are rather expensive and have a high rate of breakdown. Because of these facts the usual policy of fixed lot-sizes is not the best policy. The following problems require to be solved:

(a) How to reduce costs with a simple rule by combining breakdown and changeovers?

(b) How to balance numbers of spare tools and reorder levels, to avoid an undesirable number of stock-outs?

The first problem can be solved mathematically. By Monte Carlo methods the second problem can be solved by simulating the process, applying the production rule found in the first part.

2. THE MODEL OF THE PROCESS FOR ONE ITEM

For the first part of the research the following assumptions are made:

(a) Demand is constant.

(b) The lead time for an order is constant.

(c) Spare tools are available at any moment. At a later stage of the research this statement will be reconsidered. As a consequence of assumptions (a), (b) and (c), reorder levels can be assumed fixed in such a way that new series are started at exactly zero stock level.

(d) The probability of a breakdown of a tool is independent of the time, so the number of products made between two breakdowns has a negative exponential distribution.

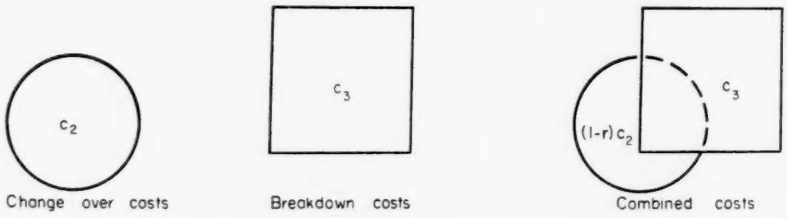
(e) Inventory costs are proportional to the average inventory level.

(f) If the production is switched over to another product at a breakdown, the total costs of breakdown and changeover are less than when they take place separately. Let the average breakdown costs be c_3 and the average changeover costs c_2, then, on combining, the total costs will be

$$c_2' + c_3 = (1-r)c_2 + c_3$$

where $$0 \leqslant r \leqslant 1.$$

This principle of overlapping costs is shown in Figure 1. A simple numerical example may serve to illustrate this point:



	Breakdown	Changeover	Combination
Repair costs	50	—	50
Set-up costs	5	20	20
Clerical costs	4	10	10
Total costs	$c_3 = 59$	$c_2 = 30$	$c_2' + c_3 = 80$

Therefore

$$r = \frac{59 + 30 - 80}{30} = 0{\cdot}3$$

in this case.

(g) The following rule is introduced for the quantities to be produced:

Changeover to another product either at the first breakdown after at least a number of Q_1 units has been produced or when a number of Q_2 units is reached ($Q_2 > Q_1$).

(h) Time needed for changing tools is neglected.

3. THE COST FUNCTION

It is the cost function that has been minimized; it may be defined by K = expected relevant costs per unit produced.

3.1. *Inventory Costs*

Q = lot-size,
s = demand per unit of time,
p = production per unit of time,
c_1 = inventory costs per unit stored per unit of time,
K_1 = expected inventory costs per unit produced,
$E(..)$ = expectation of ...

The production of a particular lot-size Q covers a period of Q/s units of time (see Figure 2).

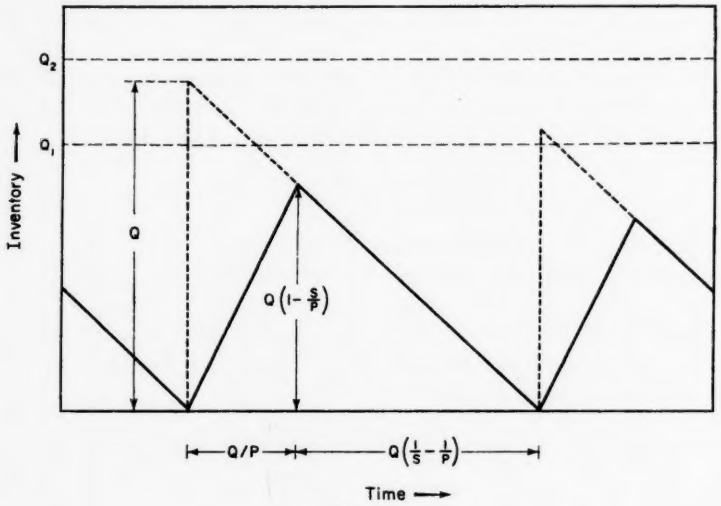


FIG. 2.

The average inventory during that period is

$$\tfrac{1}{2}Q\left(1-\frac{s}{p}\right).$$

So the inventory costs over that period are equal to

$$\frac{Q}{s}\cdot\tfrac{1}{2}Q\left(1-\frac{s}{p}\right)\cdot c_1=\tfrac{1}{2}Q^2\left(\frac{1}{s}-\frac{1}{p}\right)c_1.$$

The expected costs per unit produced for a given Q distribution can be expressed as the ratio of the expected costs per lot-size period to the expected lot-size:

$$K_1=\frac{E\{\frac{1}{2}Q^2(1/s-1/p)\,c_1\}}{E(Q)}=\frac{1}{2}\left(\frac{1}{s}-\frac{1}{p}\right)c_1\frac{E(Q^2)}{E(Q)}. \qquad (1)$$

3.2. *Changeover Costs*

When a series ends with a breakdown of a tool, changeover costs are reduced from c_2 to $(1-r)c_2$, as defined in section 2. So the expectation of changeover costs will be

$$K_2 = \frac{\{1-P(Q_2)\}.(1-r)c_2+P(Q_2)c_2}{E(Q)}, \tag{2}$$

where

$P(Q_2)$ = the probability a lot-size $Q = Q_2$ (and so the full amount c_2 for changeover costs must be calculated),

K_2 = expected changeover costs per unit produced.

3.3. *Breakdown Costs*

Breakdown costs are charged for the full amount in every case. The probability of a breakdown is assumed to be independent of time, so the breakdown costs per unit produced will be independent of the lot-size policy

$$K_3 = \mu c_3 \tag{3}$$

where

μ = probability of a breakdown per unit produced,

K_3 = expected breakdown costs per unit produced.

3.4. *Total Relevant Costs*

The breakdown costs can be excluded from the cost function, as they are independent of the lot-size policy. So

$$K = K_1 + K_2 = \frac{\frac{1}{2}(1/s-1/p)E(Q^2)c_1+c_2-\{1-P(Q_2)\}rc_2}{E(Q)}. \tag{4}$$

Derivation of $P(Q_2)$, $E(Q)$ and $E(Q^2)$ and so of K as functions of Q_1 and Q_2 is given in Appendix 1.

3.5. *Standard Lot-size and Standard Costs*

As an alternative, and to test the efficiency of the variable lot-size system, the lot-size can be kept constant by

$$Q_0 = Q_1 = Q_2.$$

Then

$$P(Q_2) = 1,$$

$$E(Q) = Q_0,$$

$$E(Q^2) = Q_0^2,$$

and according to (4),

$$K_0 = \frac{1}{2}\left(\frac{1}{s}-\frac{1}{p}\right)Q_0 c_1+\frac{c_2}{Q_0}.$$

By making $dK_0/dQ_0 = 0$ it is found that the optimum values of Q_0 and K_0 are

$$Q_0 = \sqrt{\left[\frac{2c_2}{(1/s - 1/p)\,c_1}\right]}, \tag{5}$$

$$K_0 = \sqrt{[2c_1 c_2(1/s - 1/p)]}. \tag{6}$$

For $p \to \infty$ formula (5) becomes the well-known formula of Camp (Wilson).

In the next sections Q_0 and K_0 will be denoted as the standard lot-size and the standard costs.

4. THE OPTIMUM POLICY

The optimum values of Q_1 and Q_2 can be determined by letting

$$\frac{dK}{dQ_1} = 0,$$

$$\frac{dK}{dQ_2} = 0.$$

The mathematical derivations are given in Appendix 2. The results are here presented in the form of graphs for practical use as illustrated in sections 4.1 and 4.2.

4.1. *Saving of Costs*

The saving in the relevant costs in the optimum situation with respect to the standard costs is shown in Figure 3 by areas in which the savings

$$1 - R = 1 - \frac{K}{K_0}$$

are more than 0·01, 0·10 and 0·25.

The parameters on the axes of the graph are:

r = reduction factor of the changeover costs, defined in section 2;

λ = the ratio of the standard lot-size Q_0 to the average length of a production run between two breakdowns; in other words, λ is the expected number of breakdowns in a series with lot-size Q_0.

The graph can be an aid in deciding which lot-size policy to use.

4.2. *Optimum Lot-size Limits*

The optimum values of Q_1 and Q_2 can be taken from Figures 4 and 5 where the ratios Q_1/Q_0 and Q_2/Q_0 are presented again as functions of λ and r.

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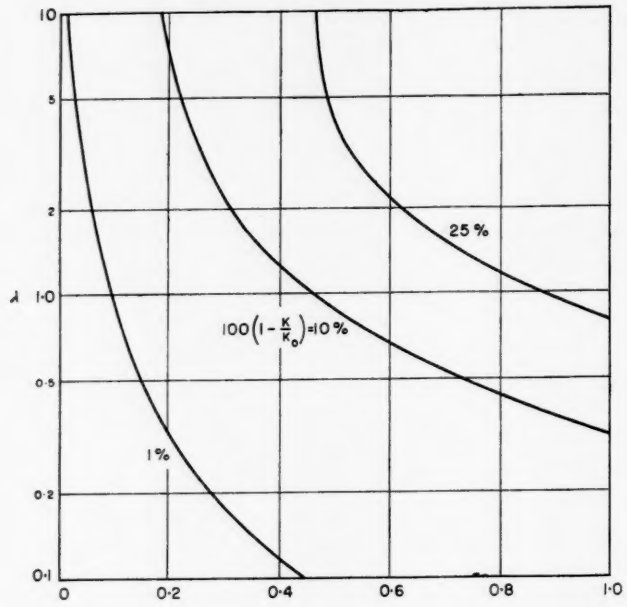

FIG. 3.

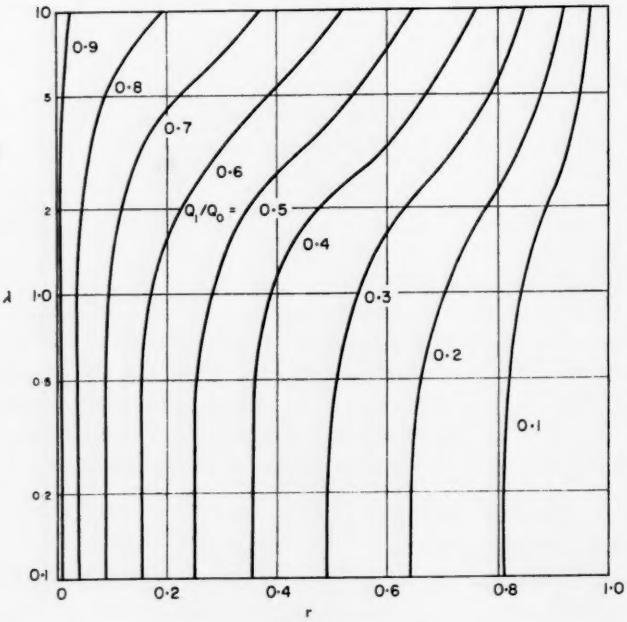

FIG. 4.

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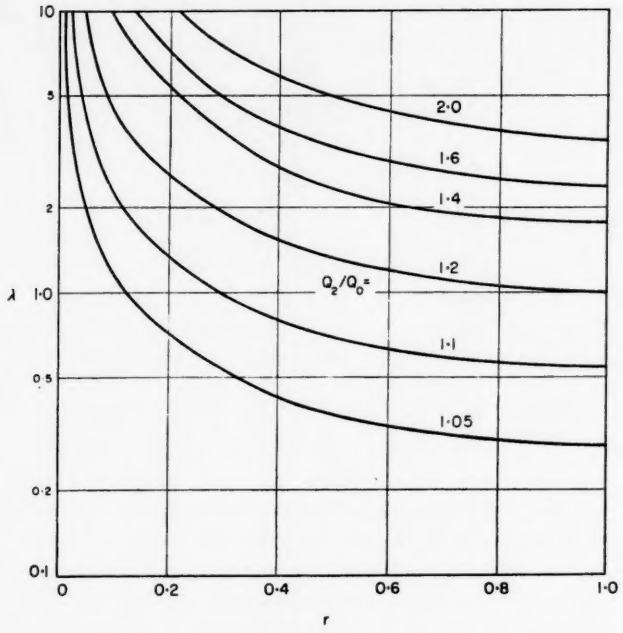

FIG. 5.

4.3. *A Practical Example illustrating the use of the Graphs*

The following data for a particular product are known:

(a) The production rate $p = 300{,}000$ units/year.

(b) The demand $s = 50{,}000$ units/year.

(c) The inventory costs $c_1 =$ hfl. 0·10/unit/year.

(d) The changeover costs $c_2 =$ hfl. 30.–.

(e) The overlapping of breakdown costs and changeover costs is estimated $rc_2 =$ hfl. 10–, and so

$$r = 0{\cdot}33.$$

(f) The average production run between two breakdowns is 2800 units. The distribution of runs could be assumed to be negative exponential. The calculation of Q_1 and Q_2 is performed as follows:

Step 1. The standard lot-size is computed by (5),

$$Q_0 = 6000 \text{ units},$$

which gives

$$\lambda = \frac{6000}{2800} = 2{\cdot}14.$$

Step 2. Figure 3 shows that for $r = 0{\cdot}33$ and $\lambda = 2{\cdot}14$ the variable lot-size system saves more than 10 per cent of the relevant costs with respect to the fixed lot-size system.

Step 3. From Figures 4 and 5 we obtain

$$\frac{Q_1}{Q_0} = 0{\cdot}5, \quad \frac{Q_2}{Q_0} = 1{\cdot}25,$$

and consequently,

$$Q_1 = 3000 \quad \text{and} \quad Q_2 = 7500.$$

The rule for this product must be:

Stop production and changeover to another product either at the first breakdown after 3000, or when 7500 units have been produced.

5. THE EFFECT OF A LIMITED SUPPLY OF TOOLS, INVESTIGATED BY MONTE CARLO METHODS

The mathematical model, on which the previous sections are based, assumes that tools are available at any time to start production if the reorder level is reached, or to change tools if a tool breaks down before an amount Q_1 has been reached. In practical cases one has to take a possible non-availability of tools into account. In the first place, the reorder level and the number of tools should be balanced in such a way that the probability of stock-outs is as small as required. In the second place, the optimum values of Q_1 and Q_2 should be reconsidered. For this purpose it was necessary to introduce new elements into the model, which became too complicated for mathematical analysis.

A Monte Carlo programme was designed for an IBM650. Except for assumption (c) the model as given in section 2 could be maintained with the following additions:

(i) The repair time of a tool (repair time = waiting time + service time) has a negative exponential distribution.

(j) In case of a stock-out, orders are not cancelled. An important consequence of this assumption is that the simulation can be performed without introducing the reorder level, for the stock can be computed with regard to the reorder level. At the end of the simulation the probability of stock-outs can be computed as a function of the reorder level.

The main flow chart is shown in Appendix 3.

The input variables needed for the simulation are:

(a) Demand per day.
(b) Production per hour.
(c) Average production run between two breakdowns.
(d) Average repair time in hours.
(e) Number of working hours per day for production shop and for repair shop.

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(f) Number of tools.

(g) Number of machines available; this number is normally one, but in cases of high demand the simultaneous use of more machines for one item may be required.

(h) Lot-size limits Q_1 and Q_2, which may be calculated with help of the derived theory.

(i) Information about length and number of simulation periods.

The output of the programme contains the following information:

(a) Reorder levels required for fixed probabilities of being stock-out.

(b) Average length of stock-out periods, average number of products delivered too late, etc.

Because of the simple stochastic behaviour of the variables, the time steps within the simulation programme could be made according to the discontinuous points in the process (a series starts or ends, a tool breaks down or comes back, etc.).

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Depending on the input variables for a particular case, the simulation time for one year varied from 2 min up to 10 min.

Because of a high variability of the process variables and the costs, the problem had to be considered for each item separately. Solutions could not be given in the form of graphs or tables for general purposes.

6. GENERAL CONCLUSIONS

From the Monte Carlo data the following conclusions could be drawn for the cases under notice:

(a) Because of a high variance of the stock-out time per year, numbers of tools and reorder levels should be balanced in a way that the probability of going stock-out is very low.

(b) A considerable decrease in the number of tools can be obtained by decreasing the repair times. Introduction of a priority system in the repair shop can be advocated such that tools for items with a high rate of demand and with a high breakdown rate have the highest priority.

(c) The lot-size limits, Q_1 and Q_2, derived under the assumption that spare tools are available at any time, can be considered optimum for practical purposes.

APPENDIX 1

Probability Distribution of the Lot-size Q

According to the lot-size decision rule, the probability that a lot-size is smaller than Q_1 or larger than Q_2 is zero. The probability function between these limits, including the upper limit Q_2, depends on the behaviour of the runs between breakdowns. This behaviour is assumed to be negative exponential. So the probability of a breakdown is independent of time, say μ in unit production time.

The probability function of the lot-size Q can be stated as follows:

$$\left.\begin{aligned} F(q) &= 0 && \text{for} \quad q \leqslant Q_1, \\ F(q) &= 1-e^{-x} && \text{for} \quad Q_1 \leqslant q \leqslant Q_2, \\ F(q) &= 1 && \text{for} \quad Q_2 < q, \end{aligned}\right\} \tag{8}$$

where

$F(q) = P(Q<q)$ = the probability that a lot-size is smaller than q,

$$x = \mu(q-Q_1).$$

The function $F(q)$ is discontinuous for $q = Q_2$. The probability that the lot-size is exactly Q_2 is

$$P(Q_2) = P(Q = Q_2) = 1-F(Q_2) = e^{-b}, \tag{9}$$

where $b = \mu(Q_2-Q_1)$. Further, let $a = \mu Q_1$, then

$$E(Q) = \int_{Q1}^{Q2} q\,dF(q)+P(Q_2)\,Q_2 = \frac{1}{\mu}(a+1-e^{-b}), \tag{10}$$

$$E(Q^2) = \int_{Q1}^{Q2} q^2\,dF(q)+P(Q_2)\,Q_2^2$$

$$= \frac{1}{\mu^2}(a+1-e^{-b})^2+1-2b\,e^{-b}-e^{-2b}. \tag{11}$$

Substituting (9), (10) and (11) into the cost function, as stated by (4), introducing the parameter $\lambda = \mu Q_0$ where

μ = breakdown probability for the production of one unit, and
Q_0 is stated by (5),

and dividing the cost function by the standard costs (6), the cost ratio R is obtained

$$R = \frac{K}{K_0} = \frac{\lambda(1-r+r\,e^{-b})+\lambda^{-1}\{a^2+2a+2-2(a+b+1)\,e^{-b}\}}{2(a+1-e^{-b})} \tag{12}$$

APPENDIX 2

Optimum Values of Q_1 and Q_2

The optimum conditions stated by (7) are identical with

$$\frac{dR}{da} = 0,$$

$$\frac{dR}{db} = 0, \tag{13}$$

where

$$R = \frac{K}{K_0}$$

is given by (12),

$$a = \mu Q_1,$$

$$b = \mu(Q_2 - Q_1).$$

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After performing the differential operations and some simplifications of the equations obtained, it is found that (13) can be replaced by

$$a^2 + 2a(1-e^{-b}) = (1-r)\lambda^2,$$

$$2(b-1+e^{-b}) = r\lambda^2. \tag{14}$$

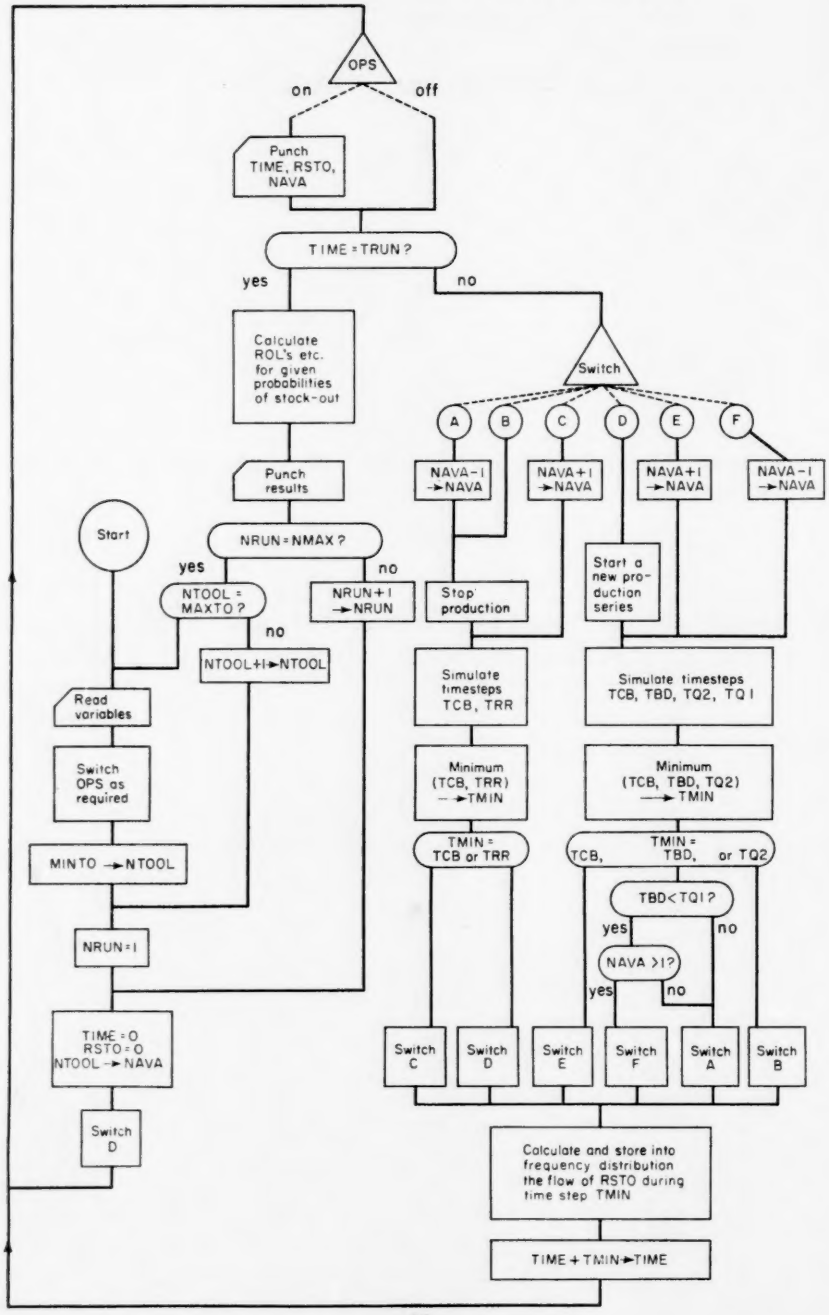
The optimum values of a and b can be calculated if r and λ are given.

The ratios Q_1/Q_0 and Q_2/Q_0 are then known, for

$$\frac{Q_1}{Q_0} = \frac{a}{\lambda},$$

$$\frac{Q_2}{Q_0} = \frac{a+b}{\lambda}. \tag{15}$$

Substituting the optimum values of a and b into (12) will give the optimum cost ratio R.


OPS
on
off
Punch TIME, RSTO, NAVA
TIME = TRUN ?
yes
no
Calculate ROL's etc. for given probabilities of stock-out
Switch
A
B
C
D
E
F
Punch results
NAVA−1 →NAVA
NAVA+1 →NAVA
NAVA+1 →NAVA
NAVA−1 →NAVA
Start
NRUN = NMAX ?
yes
no
NTOOL = MAXTO ?
NRUN+1 →NRUN
Stop production
Start a new production series
no
NTOOL+1→NTOOL
Simulate timesteps TCB, TRR
Simulate timesteps TCB, TBD, TQ2, TQ1
Read variables
Switch OPS as required
Minimum (TCB, TRR) →TMIN
Minimum (TCB, TBD, TQ2) →TMIN
MINTO →NTOOL
TMIN = TCB or TRR
TMIN = TCB, TBD, or TQ2
TBD<TQ1 ?
yes
no
NAVA >1?
yes
no
NRUN=1
TIME=0 RSTO=0 NTOOL→NAVA
Switch C
Switch D
Switch E
Switch F
Switch A
Switch B
Switch D
Calculate and store into frequency distribution the flow of RSTO during time step TMIN
TIME+TMIN→TIME

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APPENDIX 3

FLOW CHART for Monte Carlo program "OPTIMAL NUMBER OF TOOLS"

GLOSSARY to chart on p. 52

MINTO = minimum number of tools
MAXTO = maximum number of tools
NTOOL = actual number of tools
NAVA = actual number of tools available
NRUN = number of the actual run
NMAX = number of runs
TIME = actual time
TMIN = time step
TRUN = time per run
TCB = time till a tool will come back from repair shop
TBD = time till a tool will break down
TRR = time till the reorder level will be reached
TQ1 = time till an amount Q1 will be produced
TQ2 = time till an amount Q2 will be produced
ROL = reorder level
RSTO = stock related to reorder level
OPS = optional punch switch

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Elementary Theory of Optimal Silo Storage Design

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This study indicates a method for calculating an economic division of warehouses into compartments when different varieties have to be stored simultaneously and separately. The study deals with cases where the total storage capacity demand and the number of varieties are known and fixed, but where the storage capacity demand of each variety is unknown. Problems of this nature exist in the preliminary designing of grain silos, fleets of transport vehicles, utensils in public kitchens, etc. If the warehouse capacity can cope with the total storage capacity demand, but the number of compartments is insufficient, it is possible that part of the material to be stored cannot be accommodated, although certain compartments may only be partly filled; this will be due to the impossibility of storing different varieties together. The materials will have to be stored in other warehouses, at a higher charge per capacity unit than that of the warehouse in question.

The division of the warehouse into a large number of compartments will assure storage of most, if not all, the material. On the other hand, increase in the number of compartments (and installations) makes for increased investment in transport installation, partitions and additional storage capacity to compensate for space taken up by these partitions and installations.

The object of this study is to introduce a method for calculating the optimal division of a warehouse into compartments, so that the annual costs of storage and capital investment are minimized.

It is possible—by introducing not very restrictive statistical assumptions—to calculate the annual costs of storing excluded material elsewhere for different modes of partitioning a warehouse into compartments. The annual building cost of the warehouse is calculated in the ordinary way.

In this study a solution is offered for the partitioning of a warehouse into equal compartments. It is to be expected that the extension of this solution to cases of the partition into compartments, differing from each other as to storage capacity, will tend to lower costs.

Cette étude indique une méthode de calcul d'une division économique des entrepôts en compartiments où l'on puisse entreposer simultanément et séparément différentes variétés. L'étude porte sur des cas où la demande de volume total d'entreposage et le nombre de variétés sont connus et fixes, mais où la demande de volume d'entreposage de chaque variété est inconnue. Des problèmes de ce genre se posent dans la conception préliminaire des silos à céréales; parcs de véhicules de transport; ustensiles des cuisines publiques.

Si le volume de l'entrepôt suffit à répondre à la demande totale de volume d'entreposage, mais si le nombre des compartiments est insuffisant, il est possible qu'une partie des matières entreposées ne trouve pas de place, alors que certains autres compartiments ne seront que partiellement remplis; cela résulte de l'impossibilité d'entreposer différentes variétés ensemble. Il faudra entreposer les matières en question dans d'autres entrepôts, à un prix plus élevé par unité de volume que celui de l'entrepôt en question.

La division de l'entrepôt en très grand nombre de compartiments assure l'entreposage de la majeure partie, sinon de toutes les matières à entreposer.

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Par contre, l'augmentation du nombre de compartiments (et leur installation) exige un investissement accru dans les moyens de transport; les cloisonnements et le volume additionnel de l'entreposage nécessaire pour compenser l'espace occupé par les cloisonnements des installations en question.

Le but de cette étude est de présenter une méthode de calcul de la division optimum d'un entrepôt en compartiments, afin de réduire au minimum les coûts annuels de l'entreposage et les investissements de capitaux.

En admettant des hypothèses statistiques qui ne sont pas très restrictives, il est possible de calculer le coût annuel de l'entreposage, des matières qui se trouvent ailleurs pour les divers cloisonnements d'un entrepôt en compartiments. Le coût annuel de construction de l'entrepôt est calculé de la manière ordinaire.

Dans cette étude, on offre une solution pour la division d'un entrepôt en compartiments égaux. Il faut s'attendre à ce qu'en étendant cette solution à des cas où la division en compartiments différents les uns des autres par rapport à leur volume d'entreposage, les coûts peuvent être baissés.

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INTRODUCTION

A SILO is a warehouse serving as a container for the storage of loose materials, especially grain. In general, several varieties of loose materials have to be stored in the silo at the same time; since these varieties must be accommodated separately, the silo has to be subdivided by permanent partitions into compartments. Now the volume stored of any given variety is not a constant, but rather a function of time; for instance, the volume of corn in the silo during a winter is typically different from that stored in summer and, furthermore, the quantity needed in one winter may be different from the quantity required (for storage) during some other winter. The silo process is thus realistically described as a flow of loose material—partitioned into several varieties—through a warehouse containing many compartments subject to the following restriction: Two (or more) different varieties must not be held simultaneously in the same compartment.

The following problem arises in connection with the silo process. Is the number of compartments in the silo and is the volume of a compartment such that the requirements posed by the silo process are optimally met? In order to exemplify the problem we shall discuss two extreme possibilities of silo design, that is, the determination of (a) the number of compartments, and (b) the volume of a compartment, which typically do not represent optimal solutions. One extreme approach would be to build a silo possessing the same number of compartments as there are varieties of loose material present in the system and to designate a definite compartment to each variety. The volume of each compartment is then determined by the uppermost value of the volume requirement of the corresponding variety. This possibility of a silo design is not likely to be of optimal character, since most of the time most of the silo compartments are practically empty and not utilized. One is intuitively assured that better silo designs may be attained on increasing the number of compartments and decreasing the volume of each compartment. It is also obvious, however,

that the other extreme possibility of a silo being partitioned into a very large number of compartments each possessing a very small volume cannot be optimal either, since some of the building and maintenance costs are proportional to the number of compartments. Using the above argument, it can be seen that the optimal solution will be found somewhere between the two extreme solutions. We may summarize the situation as follows: Any reasonable loss function will contain (at least) two cost terms: (1) an increasing function of the total silo volume, and (2) an increasing function of the number of compartments in the silo. If the restriction is imposed that the loose material in the system must be accommodated, any decrease in the total volume of the silo can only be brought about by some increase in the number of compartments. The optimal solution possesses the property that in its neighbourhood the cost of increasing the number of compartments is identical with the gain in the associated decrease of total volume.

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It is possible to refine further the description of the silo process and the relevant costs. In the previous paragraphs we have made the rather unrealistic restriction of the necessity to accommodate all the material inside the silo. In any real silo process, periods will occur during which shortage of available space will cause some of the material to be stored outside the silo; this storage outside the silo is generally associated with additional cost. The impossibility of obtaining 100 per cent accommodation in the silo may be examined under various assumptions of policy and/or cost, three of which are:

(1) It is desired to accommodate all the material most of the time; only during a small fraction of time α some material may be stored outside the silo.

(2) It is desired to store the bulk of the loose material inside the silo; however (on the average) a small fraction β of the material may be stored outside the silo.

(3) A cost which is assumed to be proportional to the quantity stored outside the silo is incorporated in the loss function.†

THE SILO PROCESS

Nothing has as yet been said about the characteristics of the silo process. Indeed, there exists a wide range of possibilities to characterize this process and a general, verbal description appears to be not out of place.

The silo is being charged with material—either continuously or intermittently; this input may then reasonably be described as a stochastic process possibly

† It is trivial to add that, if the viewpoint of cost assumptions set (3) is taken, the cost of storing unit volume of material during unit time outside the silo must be larger than the cost of storing unit volume of material during unit time inside the silo; clearly, if this is not the case the solution of the problem is simply not to build a silo and to store all the material outside.

possessing a deterministic component, as for instance, a seasonal variation. In fact, if we assume k varieties to be stored in the silo we are concerned not with one, but rather with k (typically correlated) random input processes. These stochastic processes may be continuous in time or, alternatively, may occur discontinuously at some periods.

In addition to these k input processes there exist k output random processes describing the discharge of material from the silo. Assumptions of the type that were made regarding the character of the input process apply also to the output, but complex combinations are quite feasible. Thus, for instance, the ith variety may accumulate in the silo by means of a continuous random function and be discharged at discontinuous periods (not necessarily regularly spaced) and the instantaneous discharge may be represented by a continuous random variable.

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The theory of the silo process and the problems associated with it form a part of the general theory of stochastic processes and its applications; silo theory is closely connected with inventory and queueing theory. Indeed, in some sense there exists an analogy between the silo problem and the queueing problem: The number of compartments in the silo accommodating the loose material is analogous to the number of servers taking care of incoming customers, though in the (usual) queueing systems either zero or one customer are being attended to by a server at any given time, while a compartment may serve—that is, store—any volume of loose material that is equal to, or smaller than, its own volume.

However, while the points of analogy between silo theory and other established theories must be mentioned, it is important to realize that a measure of divergence also exists between these theories. The basic difference is the following: The analogue of the queue (or waiting line) in silo theory is the material stored outside the silo; in contradistinction to queueing theory, there is no additional waiting involved if some loose material is stored outside the silo. The only effect of outside storage is additional cost, whereas the flow of material through the system is not affected.

The latter property enables us to proceed to simplifications of our model. Our interest is not centred on the dynamic input-output flow characteristics of the system; rather we are concerned with a static property, the total quantity of loose material which it is required to store at an instant of random observation.† *The dynamic aspect of the model is needed only to ensure that this quantity is a random variable.* This random variable which we shall designate by U is actually a sum of k random variables—the instantaneous volume values of the k varieties required to store. Now generally these k values are correlated, since—

† The notion of random observation may be interpreted in several ways. One possibility is to assume the existence of a "random instant observation process" such that the inter-observation time possesses a negative exponential distribution.

to present a simple example—in periods of wheat storage, corn provisions are depleted and vice versa.

It is plausible to assume that the random variable U is normally distributed. If the component random variables u_i $(i = 1, 2, \ldots, k)$ are normally distributed, then from normal theory it is known that $U = \sum_{i=1}^{k} u_i$ possesses a Gaussian distribution even if the u_i's are correlated and thus dependent. Even if the u_i's are not normally distributed, the sum U frequently possesses a nearly normal distribution by virtue of some central limit theorem which is operative even under conditions of dependence and correlation between the u_i's.

Another assumption that will be made throughout this study is that the number of varieties k in the system is a constant. In many cases k is a random variable rather than a constant; however, if its variability around the expected value is small, final results are hardly affected by the simplification.

Further assumptions—pertaining to both storing characteristics and cost—will be stated on developing the models in detail.

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SOME BASIC RELATIONS

We consider U—the total volume which it is required to store—to be a normally distributed random variable which possesses an expected value V and a variance σ^2. Both of these process parameters are known and must be utilized in the silo design. The volume U has to be accommodated—under constraints described before—in a silo of m compartments. This study is concerned only with those silo solutions in which all compartments have similar properties; in particular, all compartments possess the same volume w, and the total volume of the silo W is then equal to mw. A solution of the silo problem is a specification of m and of w, such that in some sense an optimum is attained.

Now the actual number of compartments n_i required (at a random instant) for the ith variety equals

$$n_i = \left[\frac{u_i}{w}\right] + 1 \tag{1}$$

where [ ] is the bracket function, that is, $[a]$ is the largest integer not exceeding a. The first term on the right-hand side of (1), $[u_i/w]$, represents the number of required compartments for the ith variety which are completely filled; the additional compartment—described by the second term—is only partially filled. The volume of loose material in this last compartment is a random variable r_i, and it seems plausible that the distribution of it is nearly rectangular under very general conditions. A proof is given in the appendix that near-rectangularity of r_i holds if the distribution of u_i is normal and if σ_i—the standard deviation of the ith variety—is of the order of w or larger. Arguments of plausibility lead to two further results: (a) Mutual near-independence of the r_i's even

though the u_i's may be mutually dependent and correlated; (b) Near-independence of Σr_i and Σn_i. The argumentation hinges on the fact that the bulk of the ith variety is stored in (n_i-1) full compartments. On splitting up u_i into $w(n_i-1)$ and r_i (and u_j into $w(nj-1)$ and r_j) the major part of the correlation between u_i and u_j is transformed into the correlation between n_i and n_j.

The following relation holds

$$u_i = \left[\frac{u_i}{w}\right] w + r_i. \tag{2}$$

The total number of compartments required at a random instant is a random variable and equals

$$n = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \left[\frac{u_i}{w}\right] + k \tag{3}$$

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and the volume U is made up of the partial volumes u_i

$$U = \sum_{i=1}^{k} u_i = w \sum_{i=1}^{k} \left[\frac{u_i}{w}\right] + \sum_{i=1}^{k} r_i = w(n-k) + \sum_{i=1}^{k} r_i. \tag{4}$$

Now the expected value and the variance of the random variables r_i are known to be $w/2$ and $w^2/12$, respectively, under the assumption of rectangularity. Hence we find that

$$E\{w(n-k)\} = E\{U\} - E\left\{\sum_{i=1}^{k} r_i\right\} \simeq V - \frac{kw}{2} \tag{5}$$

and

$$\operatorname{var}\{w(n-k)\} \simeq \operatorname{var}\{U\} - \operatorname{var}\left\{\sum_{i=1}^{k} r_i\right\} \simeq \sigma^2 - \frac{kw^2}{12}. \tag{6}$$

From relations (4), (5) and (6) we find the main characteristics of the random variable n, "number of compartments required at a random instant".

(a) The form of the distribution is approximately normal. Clearly, account must be taken of n being discrete, so that continuity corrections must be introduced.

(b) The expected value of n is $V/w + k/2$.

(c) The variance of n is $\sigma^2/w^2 - k/12$.

If we denote the probability of requiring n compartments or less at a random instant as $P\{n\}$, we arrive—utilizing the properties (a), (b) and (c)—at the result

$$P\{n\} \simeq \Phi\left(\frac{w\{n - \frac{1}{2}(k-1)\} - V}{\sqrt{(\sigma^2 - kw^2/12)}}\right) \tag{7}$$

where $\Phi(z)$ is the (cumulative) unit normal distribution.

LOSS FUNCTIONS AND OPTIMIZATION

In the introduction we have outlined three different avenues of approach to optimization. In all of them two different costs were envisaged: (a) A term

appears in the loss function which is proportional to the number of compartments m in the silo (independent of the size of the compartment); the source of this cost is rent and interest on installations, partitions, etc., as well as current expenses associated with the running and maintenance of compartments. We designate the loss incurred by one compartment in unit time as C_1. (b) A second term appears in the loss function which is proportional to the total volume W of the silo; the source of this cost is, of course, rent and interest on investment dependent on volume, such as building materials, working time, etc. We designate the loss incurred by unit volume in unit time as C_v.

In the third cost structure, account is taken of losses associated with outside storage of material. The loss due to outside storage of unit volume in unit time will be designated as C_0.

We are now in a position to survey our three sets of alternative cost assumptions:

Cost structure I. The loss function L is represented by

$$L = mC_1 + WC_v = m(C_1 + wC_v). \tag{8}$$

We have imposed the constraint

$$P\{m\} = P\{n = m\} = 1 - \alpha, \tag{9}$$

so that by (7) the following holds,

$$\frac{w\{m - \frac{1}{2}(k-1)\} - V}{\sqrt{(\sigma^2 - kw^2/12)}} = z_{1-\alpha}. \tag{10}$$

In relation (10), $z_{1-\alpha}$ is the value that is exceeded by a unit normal variate with probability α.

Equations (8) and (10) completely determine the solution. A practical procedure for numerical computation is to establish pairs of m and w compatible with (10); insertion of these trial values in (8) leads to the optimal solution.

Cost structure II. The loss function L is again represented by relation (8). However, the constraint of storing outside the silo (on the average) a fraction β of the material needs a little further mathematical elaboration.

Suppose that at a random instant a number n of compartments is required such that $m < n \leqslant m + k$. Since there are only m compartments in the silo, material of $n - m (\leqslant k)$ compartments is stored outside. The question arises as to the quantity of material associated with these "missing" compartments. We shall make two assumptions in order to settle this problem: (1) The volume of the material of the ith variety stored outside the silo is smaller than w—to be more exact, the probability of its being w or larger is zero. (2) The $(n-m)\, r_i - s$ stored outside the silo are taken at random from the group of $k\, r_i - s$.

These assumptions are quite realistic and we are then assured that the average volume of material stored outside the silo equals $[(n-m)/2]\, w$ for the case

$0<n-m\leqslant k$. For the very rare case $n>m+k$ the average volume of material in outside storage is equal $\frac{1}{2}kw+(n-m-k)w$, or $w(n-m-\frac{1}{2}k)$.

Let the probability of requiring exactly n compartments be denoted by $p\{n\}$; also let

$$P^*\{n\} = 1-P\{n-1\} = \sum_{\nu=n}^{\infty} p\{\nu\} \tag{11}$$

describe the distribution function (cumulative to the right) of n. We put the average quantity stored outside the silo equal to βV

$$\beta V = w/2(p\{m+1\}+2p\{m+2\}+\ldots kp\{m+k\} \\ +(k+2)p\{m+k+1\}+(k+4)p\{m+k+2\}+\ldots). \tag{12}$$

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Further manipulation yields

$$\frac{2\beta V}{w} = \sum_{\nu=1}^{\infty} P^*\{m+\nu\}+\sum_{\nu=1}^{\infty} P^*\{m+k+\nu\}. \tag{13}$$

The terms appearing on the right-hand side of (13) are approximated by (7). Thus,

$$\frac{2\beta V}{w} \simeq \sum_{\nu=1}^{\infty} \Phi\left(\frac{V-w\{m+\nu-\frac{1}{2}(k+1)\}}{\sigma^2-kw^2/12}\right), \\ +\sum_{\nu=1}^{\infty} \Phi\left(\frac{V-w\{m+\nu+\frac{1}{2}(k-1)\}}{\sigma^2-kw^2/12}\right). \tag{14}$$

Numerical computations are not too simple, but are not beyond the reach of a simple desk calculating machine.

Cost structure III. In this case the loss function is more complex

$$L = mC_1+mwC_v+\frac{w}{2}\left(\sum_{\nu=1}^{\infty} P^*\{m+\nu\}+\sum_{\nu=1}^{\infty} P^*\{m+k+\nu\}\right)C_0. \tag{15}$$

The solution is obtained by setting

$$\left(\frac{\partial L}{\partial w}\right)_m \text{ and } \left(\frac{\partial L}{\partial m}\right)_w \text{ equal to zero.}\dagger$$

This is by no means a simple task. However, in any given problem—defined by V, σ^2, k, C_1, C_v, and C_0—judicious use may be made of trial calculations performed directly on (15). It is helpful to remember that—from purely physical considerations—a relation of the form

$$W = mw = V+\tfrac{1}{2}kw+\Delta \tag{16}$$

must hold, where Δ is of the order σ.

† Differentiation with respect to m should not be too disturbing; the solution thus obtained—in terms of non-integral m—is appropriately rounded and corrected.

DISCUSSION

The basic problems are, of course, the validity and range of applicability of our methods and models. The user of these methods must satisfy himself that

(i) a silo process exists giving rise to random variables u_i and U,

(ii) the number of varieties k is fixed (at least in some approximate sense) and known,

(iii) a nominal value V representing the average total loose materials volume in the silo—a quantity likely to be based on a policy decision—is known,

(iv) the fluctuation of the loose materials volume in the silo is known in terms of a standard deviation σ,

(v) costs as expressed by the quantities C_1, C_v, and C_0 are identifiable and available,†

(vi) costs C_1, C_v, and C_0 (as an alternative for C_0—either α or β) exhaust the cost structure of the problem.

If these assumptions are a faithful representation of reality and/or policy and applications of the proposed methods are meaningful, one has to examine more technical questions, such as validity of the normal approximation, modes of computation, etc. Also, the solution (m, w) will have to be discussed in retrospect, since some of the technical assumptions are based on inequalities involving both process parameters and values determined by the solution. Thus, for instance, one assumption, made to ensure near-rectangularity of the r_i density, is that σ_i is of the order of w or larger; the solution (m, w) obtained has to be re-examined and existence of this inequality must be established or, alternatively, appropriate conclusions must be drawn if the inequality is not satisfied.‡

While the development and discussion of our problem was given in silo terms, it must be stressed that the same type of problem arises whenever partition into compartments and containers is considered under the conditions similar to those in a silo. Containers may serve different materials at different times; simultaneous service of different materials by one container—mixing—is not allowed. Similar application may be considered when a fleet of transport vehicles is planned, when utensils in public kitchens are designed, etc.

The methods proposed here for the solution of the silo problem relate to the partition of a warehouse into equal compartments. It can be expected that partition methods envisaging different sizes of compartments may lower costs. Thus, the solution obtained in this paper serves as a first approximation

† An implicit assumption made in this study is that all varieties of loose material are similar with respect to cost, e.g. the cost of storing unit volume outside the silo is independent of the variety.

‡ Perusal of the appendix shows that even for the case $w = 2\sigma$ deviations from rectangularity are not serious.

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in the design of a silo. Obviously, structural and geometrical considerations may play an important part in the final solution. A number of actual silo designs were successfully carried out utilizing the methods described here.

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APPENDIX

Proof of near-rectangularity of the r_i density if the underlying variate is normally distributed. Consider a normal variate x possessing expected value μ and variance σ^2. It is desired to find the density of a derived variate r, defined by

$$r = x - \left[\frac{x}{w}\right] w \tag{A.1}$$

where w is some positive constant. Clearly r takes values between 0 and w.

Let the density be represented by $\psi(r)$. We find it by summing $f(r+jw)$ over all integers j, where f is the normal density under consideration

$$\begin{aligned}\psi(r) = \sum_{j=-\infty}^{\infty} f(r+jw) &= \frac{1}{\sqrt{(2\pi)}\,\sigma}\exp\left(-\frac{r^2}{2\sigma^2}\right) \\ &+ \frac{1}{\sqrt{(2\pi)}\,\sigma}\exp\left\{-\frac{(r+w)^2}{2\sigma^2}\right\} + \frac{1}{\sqrt{(2\pi)}\,\sigma}\exp\left\{-\frac{(r+2w)^2}{2\sigma^2}\right\} + \ldots \\ &+ \frac{1}{\sqrt{(2\pi)}\,\sigma}\exp\left\{-\frac{(r-w)^2}{2\sigma^2}\right\} + \frac{1}{\sqrt{(2\pi)}\,\sigma}\exp\left\{-\frac{(r-2w)^2}{2\sigma^2}\right\} + \ldots .\end{aligned} \tag{A.2}$$

If we set

$$\frac{r}{\sigma} = \rho, \tag{A.3}$$

$$\frac{w}{\sigma} = \omega, \tag{A.4}$$

$$\frac{r}{w} = \frac{\rho}{\omega} = \lambda, \tag{A.5}$$

$$\exp\left(-\frac{\omega^2}{2}\right) = q, \tag{A.6}$$

one can further develop $\psi(r)$,

$$\psi(r) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}\left\{\exp\left(-\frac{\rho^2}{2}\right)+\exp\left(-\frac{(\rho+\omega)^2}{2}\right)+\exp\left(-\frac{(\rho+2\omega)^2}{2}\right)+\ldots\right.$$
$$\left.+\exp\left(-\frac{(\rho-\omega)^2}{2}\right)+\exp\left(-\frac{(\rho-2\omega)^2}{2}\right)+\ldots\right\}$$
$$=\frac{\omega}{w}\frac{1}{\sqrt{2\pi}}\left\{\exp\left(-\frac{(\lambda\omega)^2}{2}\right)+\exp\left(-\frac{(\lambda\omega+\omega)^2}{2}\right)+\exp\left(-\frac{(\lambda\omega+2\omega)^2}{2}\right)\right.$$
$$\left.+\exp\left(-\frac{(\lambda\omega-\omega)^2}{2}\right)+\exp\left(-\frac{(\lambda\omega-2\omega)^2}{2}\right)+\ldots\right\}$$
$$=\frac{\omega}{w}\frac{1}{\sqrt{2\pi}}\{q^{\lambda^2}+q^{(\lambda+1)^2}+q^{(\lambda+2)^2}+\ldots q^{(\lambda-1)^2}+q^{(\lambda-2)^2}+\ldots\}$$
$$=\frac{\omega}{w}\frac{1}{\sqrt{2\pi}}q^{\lambda^2}\{1+q(q^{2\lambda}+q^{-2\lambda})+q^4(q^{4\lambda}+q^{-4\lambda})+q^9(q^{6\lambda}+q^{-6\lambda})+\ldots\}. \quad \text{(A.7)}$$

Now the definition of the theta function of the third type with arguments z and q is given by

$$\theta_3(z,q) = 1+2q\cos 2z+2q^4\cos 4z+2q^9\cos 6z+\ldots$$
$$=1+q(e^{2iz}+e^{-2iz})+q^4(e^{4iz}+e^{-4iz})+q^9(e^{6iz}+e^{-6iz})+\ldots. \quad \text{(A.8)}$$

Set

$$q^{2\lambda} = e^{2iz} \quad \text{(A.9)}$$

and the following result is obtained

$$\psi(r) = \frac{\omega}{w}\frac{1}{\sqrt{2\pi}}e^{-\omega^2\lambda^2/2}\theta_3\left(\frac{i\lambda\omega^2}{2},\ e^{-\omega^2/2}\right). \quad \text{(A.10)}$$

We modify now our notation by introducing

$$q = e^{\pi i\tau}. \quad \text{(A.11)}$$

If we consider θ_3, a function of z and τ instead of z and q, we use the notation $\theta_3(z|\tau)$. Thus

$$\psi(r) = \frac{\omega}{w}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\omega^2\lambda^2}{2}\right)\theta_3\left(\frac{i\lambda\omega^2}{2}\middle|\frac{i\omega^2}{2\pi}\right). \quad \text{(A.12)}$$

But by Jacobi's imaginary transformation,

$$\theta_3(z|\tau) = (-i\tau)^{-\frac{1}{2}}\exp\left(\frac{z^2}{\pi i\tau}\right)\theta_3\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right), \quad \text{(A.13)}$$

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so that

$$\psi(r) = \frac{\omega}{w}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\omega^2\lambda^2}{2}\right)\left(-i.\frac{i\omega^2}{2\pi}\right)^{-\frac{1}{2}}\exp\left(\frac{-\lambda^2\omega^4}{4}.\frac{-2}{\omega^2}\right)\theta_3\left(\frac{i\lambda\omega^2/2}{i\omega^2/2\pi}\middle|-\frac{2\pi}{i\omega^2}\right)$$

$$= \frac{1}{\omega}\theta_3\left(\lambda\pi\middle|\frac{2\pi i}{w^2}\right) = \frac{1}{w}\theta_3\left[\lambda\pi, \exp\left(-\frac{2\pi^2}{\omega^2}\right)\right]. \tag{A.14}$$

On expansion of the right-hand side of (A.14) we obtain

$$\psi(r) = \frac{1}{w}(1 + 2e^{-2\pi^2/\omega^2}\cos 2\pi\lambda + 2(e^{-2\pi^2/\omega^2})^4\cos 4\pi\lambda + \ldots). \tag{A.15}$$

Relation (A.15) shows that for a wide range of ω all terms except the first may be neglected and the near-rectangularity of $\psi(r)$ is thus established.

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Application of Dynamic Programming to the Control of Stock and to the Calculation of a Maximum Stock Capacity

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At the beginning the problem, as expressed by the heads of the Company which posed it, was in essence to find out whether, within the framework of a limited and well-defined expansion to level off in 1961, the existing capacity for stocks at a port used for shipping overseas would or would not be sufficient (it was thought it would not be).

If more warehouses were shown to be needed, the amount of this extra capacity was to be worked out.

It was very quickly discovered by an examination of the situation that the real problem would be rather to advise rules of stock management so as to take the best advantage of the existing stock capacity, in this way reducing the investment in buildings and maintenance to a minimum.

Study of the model confirmed that the new management rules would make it unnecessary to increase the existing stock capacity.

The model has the following characteristics: non-stationary process (e.g. an ever-changing situation), and an absence of any hypothesis on even the type of decision as to the method of renewing stock.

Dynamic Programming used for this set of circumstances has made it possible to arrive at optimum management rules, demonstrating, incidentally, the power of this method in the study of non-stationary processes.

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1. THE PROBLEM

A LARGE Company uses warehouses in a port of useful capacity $\hat{S}$ as a base for exporting by sea. Ships come to this port and load tonnage corresponding to the orders of the clients. The quantities X_i taken away during a given monthly period i, which is taken as the unit of time, vary in a random manner. The law for these shipments is normal, being well known for the past and the present. According to the Company it is possible to extrapolate safely from this information for five or six years to come.

The stores are supplied from a central depot which is some hundreds of miles from the port. Two means of transport between the central depot and the port exist, and both are used: River, the usual means of transport, less costly but having various random factors; Rail, which is more expensive but quick and sure and is therefore used on special occasions. In general, transport by rail is twice as expensive as transport by water.

The river method of transport has a number of characteristics which vary randomly, among which one of the most important is the tonnage which can be carried. This varies according to the time of year from a unit value to five

times that unit value. When the lack of barges lasts for a long time, the Company has to resort necessarily to the more costly method of transport by rail.

The question is to know if an increase in the capacity for stocks at the port during the periods when there are plenty of barges would make it possible to avoid completely or partly the use of rail transport.

The problem therefore becomes the search for an optimum capacity to stock at the port, making it possible to minimize the expected mean value of the overall cost of transport and stocks during a year.

2. ECONOMIC FUNCTION

This will include the following terms during an exercise over a year in terms of the expected mean value:

- M the annual costs relating to the eventual construction of additional warehouses;
- R the total cost of river transport per year;
- F the total cost of rail transport per year;
- D the extra cost that would result from extra handling of goods brought about by putting into stock or taking out of stock;
- A the annual cost of capital represented by the value of goods in stock.

Finally, there is a supplementary term: the penalty for going beyond the stock capacity. When customers do not order in sufficient quantity warehouses can become full, and quantities which continue to arrive by barge down the river must be stocked in rented warehouse accommodation. The probable extra cost of this penalty for such extra costs for a year is called P.

It now becomes necessary to examine these terms as functions of $\hat{S}$, the desired capacity, introducing into the calculations the stock at the beginning of the period S_i, and reckoning the periods i in reverse order of time (unless there is some special note to the contrary).

3.1. *Construction of New Warehouses (M)*

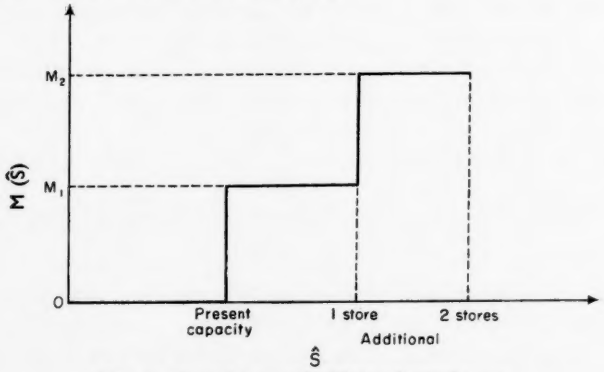

FIG. 1. *Annual costs of additional warehouses.*

The size of the handling installations, together with the strong restraining effect of the topographical factors, led the company to the conclusion that an increase in stock capacity could only be made by building one or two extra warehouses of equal size and of equal cost.

The term $M(\hat{S})$ can now therefore be represented by a discontinuous function with three levels (Figure 1), and it is easy to take into account, in the yearly costs, only the *extra costs*, in relation to the present situation.

It then looked sensible to attack the problem simply, by calculating the annual overall cost of management for three hypotheses for the value $\hat{S}$ (0, 1, or 2 supplementary warehouses), and to choose the solution giving the minimum cost.

3.2. *The Value of Merchandise in Stock* (A)

To calculate this value it did not seem correct to calculate the value of stock merchandise at its cost price at the port, because the merchandise might be stocked either at the central depot or at the port. The fact of being stocked at the port would only mean immobilizing a larger amount of capital, the value of the ton of goods by the time it had reached the port having increased by the cost of transport. The annual cost, at a given rate θ, of unit cost of transport by river r (the usual method of transport in this study), was adopted therefore as representing the value of immobilized stocks in the store warehouse. The classical hypothesis was used that the volume of stock S varied in a linear fashion during a monthly period. For the period i during which the stock varies from S_i to S_{i-1}, the costs caused by immobilization of the goods in the warehouse would be: $\frac{1}{2}\theta r(S_i + S_{i-1})$.

3.3. *Cost of Handling* (D)

The hypothesis was made and confirmed by the facts that if, during the course of the period i, the arrivals Q_i were greater than the shipments X_i, the difference was put into stock, thereby entailing an extra unit cost of handling (d), the part of the arrivals corresponding to the despatches being directly transferred from the river barges to sea-going ships. The same situation obtained with a reduction of stock if the arrivals were less than the despatches. The expected mean value of the cost of maintenance for period i was then

$$d\left[\int_{-\infty}^{0}(Q_i - X_i)f_i(X_i - Q_i)\,d(X_i - Q_i) + \int_{0}^{+\infty}(X_i - Q_i)f_i(X_i - Q_i)\,d(X_i - Q_i)\right],$$

$f_i(X_i - Q_i)$ being the frequency functions of the difference during the period

$$X - Q.$$

An overall estimate showed that about 10 per cent to 15 per cent of the goods were subject to this supplementary handling, and that this extra cost per ton was of the order of 5 per cent to 6 per cent of the cost of river transport. The incidence of the factor D was thus less than 1 per cent of the overall cost.

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3.4. *Transport by Rail (F)*

In spite of its speed, rail transport because of its high cost is considered as a penalty which is only incurred when the stock at the port has fallen to zero and ships are waiting for goods, that is to say when during the period i

$$X_i > Q_i + S_i \ (S_i \text{ stock at the beginning of the period}).$$

The procedure then is as if one brought by rail the difference $X_i - (Q_i + S_i)$ at the unit cost f.

The expected mean cost for a period is then

$$F = f \int_{S_i}^{\infty} (X_i - Q_i - S_i) f_i(X_i - Q_i)\, d(X_i - Q_i).$$

3.5. *Transport by River (R)*

We have already seen that river transport has certain random characteristics. An enquiry showed that the lack of barges was linked with various circumstances, of which the principal were the following:

(i) Lack of water in the river, which makes it necessary to lighten the barges and thus to reduce the tonnage carried by each barge.
(ii) Ice and fog, which lengthen the turn-round period of the barges.
(iii) Length of night at winter (the barges only move in the daytime), which also lengthens the turn-round period.
(iv) Competing traffic.

A detailed study of these factors, particularly the first, revealed enormous delays and very great difficulties. It was therefore decided, instead of trying to discover the causes of these physical factors on the efficiency of river traffic, to assess the consequences of these external factors.

The mechanism of loading the barges is as follows. At the beginning of the month the Company advises the freight agents of its intentions and desired transport. An examination of the relative statistics showed that at certain times of year, and particularly in the summer, these requirements were met, but in the winter there was fairly frequently a difference between the required and actual loadings. Let C_i be the tonnage required to be loaded in a month i and let Q_i be the actual tonnage loaded in the month i.

A regression study relating the deficit margin $C - Q$ and the depth of water at the embarking point was found to be fruitful, a certain depth of water in the river H_0 being found to be critical, and above this value there being no deficit value that was significant in the usual statistical sense. For the regression study all the daily levels of water were limited up to the critical level H_0, replacing all those above H_0 by H_0, and the monthly mean $\bar{H}^*$ was then calculated from these truncated values. The coefficient of correlation between the tonnage for the month i, $(C_i - Q_i)$ and the difference $H_0 - \bar{H}^* = \Delta H$ came to more than 0·70,

thus explaining more than half of the variance of the deficit, the other factors (competing traffic, ice, etc.) being responsible for the remaining variance.

Additionally, it is of interest that the mean depth of water during a given period is correlated to some extent with the depth of water for the previous period. The regression was therefore calculated between the values of the variable ΔH of a given month and of the preceding month, using statistics for the past sixty years, obtaining in this way twelve regressions of the type

$$\Delta H_{\text{April}} = a_{\text{April}} \Delta H_{\text{March}} + \epsilon^2{}_{\text{April}}\, t;$$

a represents the coefficient of regression appropriate to the month of April, ϵ^2 the residual variance for the same month, t a reduced normal variant, given the large number of different causes that can affect the depth of water in a river.

Provided with these elements of information, the Company would be able to determine the probability knowing the mean depth of water in the past month and the deficit that they would have to face in the following month.

It would be sufficient if the Company wished to transport effectively Q_i to fix its desired tonnage at the level C_i, so that

$$C_i - \bar{Q}_i = \alpha_i \Delta H_{i-1} + \beta_i, \tag{3.5.1}$$

the coefficients α_i, β_i being determined by the preceding considerations. The real value of Q_i will differ from that of $\bar{Q}_i$ by a random variable x_i with a mean of zero.

The term linked with water transport in the economic function could then be worked out under the form $r_i \bar{Q}_i$, $\bar{Q}_i$ being the mean quantity and r_i the freight expenses applicable to the period under consideration. The term R for the period i could then be written as its expected mean value

$$R = r_i[C_i - (\alpha_i \Delta H_{i-1} + \beta_i)]. \tag{3.5.2}$$

3.6. *Extra Stock* (P)

The penalty to pay when the stock at the port is above the capacity of the warehouses $\hat{S}$ takes the form of a given cost per ton per day during a minimum period of time. For the conditions of the problem, the following approximation could be considered as justified:

If the stock at the beginning of a period is $S_i > \hat{S}$, the excess $S_i - \hat{S}$ is stocked on barges and costs $c(S_i - \hat{S})$ during a period, whether the stock falls below $\hat{S}$ during the course of that period or not.

3.7. *Overall Costs for a Period* (L_i)

The introduction of two auxiliary variables makes it possible to simplify the notation as follows:

Let us call Y_i the availability for a period when the desired amounts are completely met, C_i+S_i, and Z_i the total shipments actually exported, plus deficit:

$$\left.\begin{aligned} Y_i &= C_i+S_i, \\ Z_i &= X_i+C_i-Q_i = X_i+\alpha_i\Delta H_{i-1}+\beta_i+x_i. \end{aligned}\right\} \tag{3.7.1}$$

This makes it possible, as can be seen, to combine in one random variable Z, the two random variables X and x. We shall consider as normal the random variable Z, since this is the sum of two random independent variables, of which one, X, is definitely normal and the other has a distribution not far removed from the normal distribution, as a large number of independent causes affect barge deficiency.

$g_i(Z_i)$ and $G_i(Z_i)$ will be respectively the density and the distribution function of this variable Z_i during the period i, with a mean of $\bar{Z}_i$ and a standard deviation of σ_i.

With the new variables, the term $S_i-(X_i-Q_i)$, which represents the stock at the end of the period, becomes Y_i-Z_i, and we arrive finally at the varying economic function for the period i

$$\begin{aligned} L_i(S_i) = r_i[C_i-(\alpha_i\Delta H_{i+1}+\beta_i)]+f\int_{Y_i}^{\infty}(Z_i-Y_i)\,g_i(Z_i)\,dZ_i+\dots \\ +c\int_0^{Y_i-\hat{S}}(Y_i-\hat{S}-Z_i)\,g_i(Z_i)\,dZ_i+d\left[\int_0^{Y_i-S_i}(Y_i-S_i-Z_i)\,g_i(Z_i)\,dZ_i+\dots\right. \\ \left.+\int_{Y_i-S_i}^{\infty}(S_i-Y_i+Z_i)\,g_i(Z_i)\,dZ_i\right]+\tfrac{1}{2}\theta\bar{r}(S_i+Y_i-Z_i). \end{aligned} \tag{3.7.2}$$

It is now possible to see the precise meaning of the three variables, S, Y, and Z:

S is the "state" variable which characterizes the passage from one period to another.

Z is the random variable dependent on the orders and randomness of navigation.

Y is *the* decision variable. It will give to those responsible for transport who know the stock S_i from the beginning of the period, the value C_i, which will allow them to order the optimum desired tonnage from the freight agent.

4. CHOICE OF THE PERIOD—NON-STATIONARINESS OF THE PROCESS

The fact that the Company makes forecasts of its sales, X_i, month by month, and the additional fact that contracts are made with the freight agents at the beginning of each month, have led us to choose the month as the unit period,

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and it follows that calculation has been made each month of the optimum value of the variable Y_i which, bearing in mind the state of the stock at the beginning of each month S_i, will make it possible to calculate the desired value C_i for each month.

5. THE SEQUENTIAL PROGRAMME

$L_i(S_i, Y_i)$ being the value of the economic function limited to a period i, we shall call $F_i(S_i)$ the value of the economic function cumulated for all the following periods, including the period i, $i-1, i-2, \ldots, 3, 2$, and 1 (the values of the function being discounted at α).

The fundamental dynamic programming theorem states that the minimum for F_i is obtained by minimizing the cost of the period i, on condition that management optimum rules are also worked out for the whole of the following periods. This produces then the fundamental relation

$$F_i(S_i) = \min_{Y_i} [L_i(Y_i, S_i) + \alpha \overline{F_{i-1}(S_{i-1})}], \dagger \tag{5.1}$$

which we can now write, remembering that $S_{i-1} = Y_i - Z_i$,

$$F_i(S_i) = \min_{Y_i} \left[L_i(Y_i, S_i) + \alpha \int_0^{Y_i} F_{i-1}(Y_i - Z_i)\, g_i(Z_i)\, dZ_i + \ldots \right.$$
$$\left. \ldots + \alpha \int_{Y_i}^{\infty} F_{i-1}(0)\, g_i(Z_i)\, dZ_i \right]. \tag{5.2}$$

This minimum must respect the relation $C_i \geqslant 0$, be $Y_i \geqslant S_i$, $C_i \geqslant 0$.

The minimum will be obtained for a value Y_i^* such that $\partial F_i / \partial Y_i = 0$. Let

$$\frac{\partial L_i}{\partial Y_i} + \alpha \int_0^{Y_i} \frac{\partial}{\partial Y_i} [F_{i-1}(Y_i - Z_i)]\, g_i(Z_i)\, dZ_i = 0. \tag{5.3}$$

When the process continues until the last period is under consideration and there remains a residual stock S_0, it is natural to think that the management cost for the following period, which is limited to the running out of the stock S_0, contains a term proportional to S_0 and eventually a constant term.

From this consideration let us see under what conditions the functions $F_i(S_i)$, linear in S_i, are a solution of the equation (5.2).

Let
$$F_i(S_i) = p_i S_i + F_i(0), \tag{5.4}$$

whence
$$F_{i-1}(Y_i - Z_i) = p_{i-1}(Y_i - Z_i) + F_{i-1}(0). \tag{5.5}$$

The derivative becomes

$$\frac{\partial L_i}{\partial Y_i} + \alpha p_{i-1} \int_0^{Y_i} g_i(Z_i)\, dZ_i = \frac{\partial L_i}{\partial Y_i} + \alpha p_{i-1}\, G_i(Y_i). \tag{5.6}$$

† $\overline{F_{i-1}}$ denoting the expected mean value of F_{i-1} for all possible values of S_{i-1}.

The expression (3.7.2) of $L_i(Y_i, S_i)$ given, replacing C_i by $Y_i - S_i$,

$$\frac{\partial L_i}{\partial Y_i} = r_i - f[1 - G_i(Y_i)] + cG_i(Y_i - \hat{S}) + 2dG_i(Y_i - S_i) - d + \ldots + \tfrac{1}{2}\theta\bar{r}, \quad (5.7)$$

which in the optimum condition makes it possible to calculate Y_i in view of the relation (5.3)

$$(r_i - d + \tfrac{1}{2}\theta\bar{r} - f) + fG_i(Y_i) + cG_i(Y_i - \hat{S}) + 2dG_i(Y_i - S_i) + \alpha p_{i-1} G_i(Y_i) = 0. \quad (5.8)$$

This relation has a solution Y_i^* if $r_i - d + \frac{1}{2}\theta\bar{r} - f < 0$, which is in fact true, given the importance of the term f in this case.

The second derivative is positive,

$$fg_i(Y_i) + cg_i(Y_i - \hat{S}) + 2dg_i(Y_i - S_i) + \alpha p_{i-1} g_i(Y_i) > 0,$$

all these terms being positive, the G_i being additionally monotonic increasing functions; there is therefore a *minimum* and a *unique* minimum.

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It is then possible to write, for the optimum,

$$F_i(S_i) = L_i(Y_i^*, S_i) + \alpha F_{i-1}(0) + \alpha p_{i-1} \int_0^{Y_i^*} (Y_i^* - Z_i)\, g_i(Z_i)\, dZ_i, \quad (5.9)$$

the two latter terms being independent of S_i.

Besides, $L_i(Y_i^*, S_i)$ is written

$$L_i(Y_i^*, S_i) = -r_i S_i + d \int_0^{Y_i^* - S_i} (Y_i^* - Z_i)\, g_i(Z_i)\, dZ_i + \ldots$$

$$\ldots + d \int_{Y_i^* - S_i}^{\infty} (S_i - Y_i^* + Z_i)\, g_i(Z_i)\, dZ_i + L_i(Y_i^*, 0), \quad (5.10)$$

and with this form of L_i, $F_i(S_i)$ is not linear in S_i, and one can see that the degree in S_i of F_i is that of L_i.

Calculations show that the sensitivity of Y_i^* to d is very weak and that the cost of handling is relatively small. It is thus possible to eliminate the terms in d (handling) in the expression of L_i, which gives the tremendous advantage of providing an expression which is linear in S_i for L_i and therefore for F_i.

It is then possible to write

$$L_i(Y_i^*, S_i) = -r_i S_i + L_i(Y_i^*, 0) \quad (5.11)$$

and

$$F_i^*(S_i) = -r_i S_i + F_i^*(0), \quad (5.12)$$

which makes it possible to identify

$$p_i = -r_i. \quad (5.13)$$

The equation which gives Y_i^* is therefore

$$\left(r_i + \frac{\theta\bar{r}}{2} - f\right) + (f - \alpha r_{i-1})\, G_i(Y_i) + cG_i(Y_i - \hat{S}) = 0, \quad (5.14)$$

or

$$G_i(Y_i)+\frac{c}{f-\alpha r_{i-1}}G_i(Y_i-\hat{S})=\frac{f-r_i-\frac{1}{2}\theta\bar{r}}{f-\alpha r_{i-1}}; \tag{5.15}$$

or again, using the reduced variable

$$u_i=\frac{Z_i-\bar{Z}_i}{\sigma_i} \tag{5.16}$$

and putting

$$\mu_i=\frac{c}{f-\alpha r_{i-1}} \quad \text{and} \quad \lambda_i=\frac{f-r_i-\theta\bar{r}/2}{f-r_{i-1}} \tag{5.17}$$

$$G(u_i)+\mu_i G\left(u_i-\frac{\hat{S}}{\sigma_i}\right)=\lambda_i. \tag{5.18}$$

The solution can be obtained graphically as in Figure 2 by constructing the cumulative curve $G(u_i)$, then the same curve displaced to the right of the u axis by S/σ_i and reduced by the coefficient $-\mu_i$; it is then sufficient to seek for the abscissa u_i^* where the distance between the two curves is equal to λ_i.

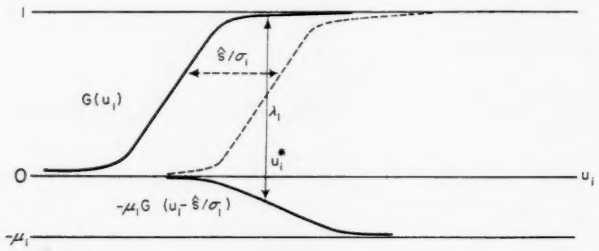

FIG. 2.

Those responsible for management must additionally calculate $\bar{Z}_i$, by adding to its export forecasts $\bar{X}_i$, the probable barge deficit $C_i-Q_i=\alpha_i\Delta H^*_{i+1}+\beta_i$, α_i and β_i being obtained from the regressions and ΔH_{i+1}, from the calculation of the mean depths of water limited to H_0 for the preceding month $(i+1)$.

6. ANNUAL COST OF MANAGEMENT

It is possible to calculate in the form of an expected mean value the probable cost of management for each period, the variable Y_i, having taken its optimum value Y_i^*. The sum over the twelve months of the year of this monthly cost will give the annual cost. This, when calculated for the various hypotheses for the value $\hat{S}$ of the capacity to stock at the port, will make it possible to decide among these hypotheses and to decide whether it is opportune or not to build one or even two extra warehouses.

The annual cost will therefore be $\Gamma(\hat{S})$:

$$\Gamma(\hat{S}) = \sum_{i=\text{January}}^{\text{December}} \{r_i[C_i-(\alpha_i \Delta H_{i+1}+\beta_i)]+f\int_{Y_i^*}^{\infty}(Z_i-Y_i^*)g_i(Z_i)\,dZ_i+\ldots$$

$$c\int_0^{Y_i^*}(Y_i^*-Z_i-\hat{S})g_i(Z_i)\,dZ_i+\tfrac{1}{2}\theta\bar{r}(S_i+Y_i^*-Z_i)\}. \qquad (6.1)$$

To this annual cost, which can be written

$$\Gamma(\hat{S}) = \sum_{\text{January}}^{\text{December}} \{r_i C_i - r_i\alpha_i\Delta H_{i+1} - r_i\beta_i + f\sigma_i g(u_i^*)+f(\bar{Z}_i-Y_i^*)[1-G(u_i^*)]$$

$$+c\sigma_i g(u_i^*-\hat{S}/\sigma_i)-c\sigma_i g\left(-\frac{\bar{Z}}{\sigma_i}\right)+c(Y_i^*-Z_i-\hat{S})\left[G(u_i^*-\hat{S}/\sigma_i)-G\left(-\frac{\bar{Z}}{\sigma_i}\right)\right]$$

$$+\tfrac{1}{2}\theta\bar{r}(S_i+Y_i^*-Z_i)\}, \qquad (6.2)$$

one must add M_1 or M_2, representing the annual cost of the one or two warehouses to be built.

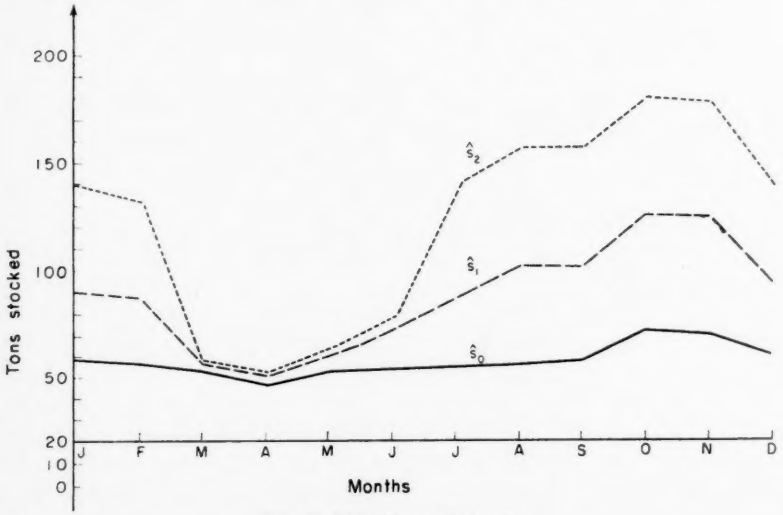

FIG. 3. *Monthly stock levels.*

By way of documentary illustration, Figure 3 shows the level of stock for the three possible hypotheses for the value of $\hat{S}$, and Figure 4 the corresponding total management cost on the basis of the hypothesis of a mean depth of water in the river obtained by examining the fluctuations in the level of the river for the last fifty years.

The graph of management cost is eloquent. The present situation $\hat{S}_0$ is definitely more advantageous than $\hat{S}_1$ or $\hat{S}_2$ in which one or two extra warehouses are built. It is evident, however, that this will only remain true so long

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as the optimum management policy represented by (Y_i^*) is in force. It will be noted that the quantities brought by rail are reduced to an extremely small amount even in the case of $\hat{S}_0$.

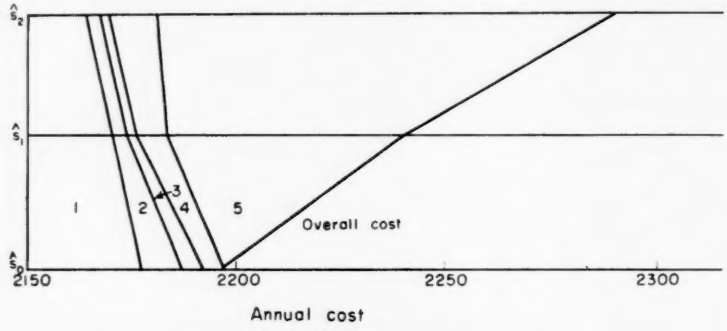

FIG. 4. *Management costs.*

7. STABILITY OF THE SOLUTION

As a mean value, the solution $\hat{S}_0$ without extra warehouses leads to costs which are much lower on average.

By way of exploring possibilities, a simulation was made using actual water levels in the river in the course of a particularly dry year. Exports in this simulation were taken as the mean forecast for each month.

The simulation on the base of this exceptional year, for the three hypotheses $\hat{S}_0, \hat{S}_1, \hat{S}_2$, leads, as one would expect, to a much higher proportion of transport by rail than for the average year previously studied and hence a much higher management cost.

It was found, however, that even in this exceptional case it would not be profitable to build extra warehouses. The solution $\hat{S}_0$ is therefore shown to be suitable and adapted to all circumstances.

8. INFLUENCE OF THE PARAMETERS

The best management policy is evidenced by the choice once a month of the value of the decision variable Y_i^*, which follows from the solution of the equation (5.14)

$$G_i(Y_i)+\frac{c}{f-\alpha r_{i-1}}G_i(Y_i-\hat{S})=\frac{f-r_i-\theta_{\bar{r}/2}}{f-\alpha r_{i-1}}.$$

It is interesting to note how the value of the Y_i^* solution is affected by the variations in the different parameters c, r_i and r_{i-1}, f and σ_i which is implicit in the function G_i.

Recalling equation (5.18), which gives u_i^*, hence Y_i^*

$$G(u_i)+\mu_i G\left(u_i-\frac{\hat{S}}{i}\right)=\lambda_i.$$

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The total differential is written

$$g(u_i)\,du_i+\mu_i g\left(u_i-\frac{\hat{S}}{\sigma_i}\right)d\left(u_i-\frac{\hat{S}}{\sigma_i}\right)+G\left(u_i-\frac{\hat{S}}{\sigma_i}\right)d\mu_i=d\lambda_i.$$

Note that

$$d\left(u-\frac{\hat{S}}{\sigma}\right)=du-\frac{d\hat{S}}{\sigma}+\frac{\hat{S}d\sigma}{\sigma^2}.$$

Knowing that

$$\mu_i\sim\frac{c}{f-r_{i-1}}$$

and

$$\lambda_i\sim\frac{f-r_i}{f-r_{i-1}}=\frac{f-r_{i-1}(1+K)}{f-r_{i-1}}\left(K=\frac{r_i}{r_{i-1}}-1\right)$$

$$d\mu=\frac{dc}{f-r_{i-1}}-\frac{c\,df}{(f-r_{i-1})^2}+\frac{c\,dr_{i-1}}{(f-r_{i-1})^2}=\frac{2}{c}\left(\frac{dc}{\mu}-df+dr_{i-1}\right)$$

$$d\lambda=K\frac{r_{i-1}\,df-f\,dr_{i-1}}{(f-r_{i-1})^2}=K\frac{2}{c^2}(r_{i-1}\,df-f\,dr_{i-1}),$$

whence

$$g(u_i)\,du+\mu_i g(u_i-\hat{S}/\sigma_i)\left(du-\frac{d\hat{S}}{\sigma_i}+\frac{\hat{S}}{\sigma_i 2}\,d\sigma_i\right)+G(u_i-\hat{S}/\sigma_i)\frac{\mu 2}{c}\left(\frac{dc}{\mu}-df+dr_{i-1}\right)$$
$$=\frac{K\mu^2}{c^2}(r_{i-1}\,df-f\,dr_{i-1}).$$

From which can be deduced:

$$\frac{du}{dc}=-\frac{\mu_i}{c}\frac{G(u_i-\hat{S}/\sigma_i)}{g(u_i)+\mu_i g(u_i-\hat{S}/\sigma_i)}$$

varies from $-0{\cdot}0003$ to $-0{\cdot}0015$

- c ↗ 10 per cent
- u ↘ 0·01 to 0·05
- Y ↘ 0·3 to 1·5 (1000 tons)

$$\frac{du}{d\hat{S}}=\frac{\mu_i}{\sigma_i}\frac{g(u_i-\hat{S}/\sigma_i)}{g(u_i)+\mu_i g(u_i-\hat{S}/\sigma_i)}$$

varies from $0{\cdot}03$ to $0{\cdot}005$

- $\hat{S}$ ↗ 10 per cent
- u ↗ 0·3 to 0·05
- Y ↗ 8 to 1·5 (1000 tons)

$$\frac{du}{d\sigma}=-\frac{\hat{S}\mu_i}{\sigma_i^2}\frac{g(u_i-\hat{S}/\sigma_i)}{g(u_i)+\mu_i g(u_i-\hat{S}/\sigma_i)}$$

varies from $-0{\cdot}02$ to $-0{\cdot}08$ for $\hat{S}_0$ actual

- σ ↗ 10 per cent
- u ↘ 0·06 to 0·25
- Y ↘ 2 to 8

$$\frac{du}{df}=\frac{\mu_2}{c}\frac{G(u_i-\hat{S}/\sigma_i)+Kr_{i-1}/c}{g(u_i)+\mu_i g(u_i-\hat{S}/\sigma_i)}$$

varies from $0{\cdot}00023$ to $0{\cdot}000005$

- f ↗ 10 per cent
- u ↗ 0·07
- Y ↗ 0·2 max

$$\frac{du}{dr_{i-1}}=-\frac{\mu^2}{c}\frac{G(u_i-\hat{S}/\sigma_i)+Kf/c}{g(u_i)+{}_i g(u_i-\hat{S}/\sigma_i)}$$

varies from $-0{\cdot}00003$ to $+0{\cdot}0004$

- r_{i-1} ↗ 10 per cent
- u ↗ 0·006 ↘ 0·06
- Y ↗ 0·18 ↘ 0·8.

Variations in u^* and hence of Y^* have the same sign as those of $\hat{S}$ and f, have a contrary sign to those of c, and have a variable sign according to the time of year with that of r_i.

Above all, it is apparent that the values of the angular coefficients are in general small, which indicates good stability for the solutions Y_i^* whatever the fluctuations of the various parameters.

It will be noted that no *a priori* policy has been taken for the control of stock and that the periodical decision obeys a single criterion of minimizing the overall cost in terms of expected mean value. It has been demonstrated by dynamic programming that the policy of raising a stock situation from a starting level of S_i up to a predetermined level Y_i^* given by a simple formula, proves to be the optimum solution when the renewal policy decision is taken at regular intervals.

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Letters to the Editor

A PROBLEM that is of considerable interest to advertisers is the allocation of a given budget over time. The theory is held that it is often a good strategy to advertise in a number of short bursts, rather than at a steady rate throughout the year.

The equation suggested by Benjamin, Jolly and Maitland in the last issue of the *Quarterly* throws some light on this problem.

The equation they proposed was:

$$R = a \log_e A + b$$

where R was the response to, and A the cost of, advertising and a and b were constants.

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Since b was found to be a negative constant this equation can conveniently be re-written as:

$$R = a \log_e \frac{A}{A_T}$$

The question of what happens when $A < A_T$ with the theoretical consequent negative response was not considered by the writers of the paper, but the rule $R = 0$ when $A < A_T$ would appear to be reasonable.

A_T then may be considered as representing a threshold level of advertising below which no response is obtained.

In these circumstances the period over which the budget is to be allocated may be considered as divided into n segments in any one of which an amount A_i (which may be zero) is spent on advertising with a consequent response

$$R_i = a \log_e \frac{A_i}{A_T}$$

The problem is then to maximize

$$\sum_{i=1}^{n} R_i$$

subject to the restriction that

$$\sum_{i=1}^{n} A_i \leqslant T$$

the total amount budgeted.

It can be shown that the optimum is achieved by advertising in units of eA_T.

This solution depends of course on the restriction

$$\frac{T}{eA_T} \leqslant n. \quad \text{If } \frac{T}{eA_T} > n$$

the optimum procedure is to spend an equal amount in each period.

If we exclude the two exceptional points from the figures given in the original paper, and take A_T as 1,230 to agree with the equation given:

$$R = 13{\cdot}991 \log_e A - 2{\cdot}841$$

it is possible to show that, adopting the above strategy, the expenditure of 3,345, i.e. 1,230*e*, on each of 6 occasions would have achieved results 3·25 per cent greater than those actually obtained with a saving in expenditure of 7 per cent.

L. J. ROTHMAN

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THE letter from L. J. Rothman, which you publish in your current issue, demonstrates just the sort of application of our model in budgetary allocation which we hoped would be made. We accept its validity and recognize its practical value. We feel gratified at having stimulated this extension which is a welcome development.

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W. P. JOLLY
J. MAITLAND*

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London, E.C.1.

OPERATIONS RESEARCH SOCIETY OF AMERICA

The 19th National Meeting will be held at the
Sheraton—Blackstone Hotel, Chicago, on 25—26 May, 1961.

APPOINTMENT

Operational Research workers will have read with interest of the appointment of Mr. Charles Hitch, immediate Past President of the Operations Research Society of America, as Assistant Secretary of Defence in President Kennedy's Administration. The Operational Research Quarterly respectfully offers its congratulations and best wishes to Mr. Hitch.

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The Application of the Negative Binomial Distribution to Stock Control Problems

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British European Airways

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Random demand and a gamma distribution of lead times results in a negative binomial distribution of the number of demands during a lead time. This fact, together with a rather surprising mathematical equality, enables one to use tables of the positive binomial distribution to determine reorder levels and also to calculate stocks of rotable components for aircraft.

IN this paper we shall consider two conceptually different but mathematically equivalent problems and obtain a solution which may be applied in practice.

The first problem is the well-known one of determining the level to which the stock of some commodity should be allowed to fall before an order for a new batch is placed. The general procedure is to choose this level so as to give a specified probability (if possible chosen for an overall optimum economic balance between stock investment and shortage costs[5]) of a stockout before the new batch arrives. This probability P_n, known as the risk level, must hence be determined as a function of the reorder level n. It will, in fact, equal the probability that the number of demands during the lead time exceeds n. Further details of the reorder level problem and methods of tackling it may be found elsewhere.[3]

The second problem will be described in the context of aircraft maintenance. In order to ensure a high standard of serviceability without incurring unnecessary aircraft delays, an airline such as BEA adopts the following replacement policy for its rotable (repairable) aircraft components:

(1) An unserviceable or life-expired component is removed from an aircraft and sent for inspection and either repair or overhaul.
(2) An immediate demand for a serviceable replacement is made to the stores and (if available) the replacement is at once fitted to the aircraft.
(3) When the previously unserviceable or life-expired component has been passed as serviceable, it is placed in the stores.

It may be seen that this process implies that the airline must have a few more of each type of component than are fitted to aircraft at any one time. The excess is known as the "float" of that type of component, and our problem

is to decide how large this float should be. One method of approach to the problem is to apply the results of queueing theory. This has been done with the somewhat restrictive assumption of random service time.[8] Such an approach may, however, prove impracticable due to the complexity of the various repair circuits and the large number of different types of component. Under these circumstances we may treat the problem in a way analogous to the reorder level problem. To do this we regard the presenting of a component for repair as a "demand" and the interval between the removal of the component and its entry into stores, as the "lead time". Provided there is no overtaking in the

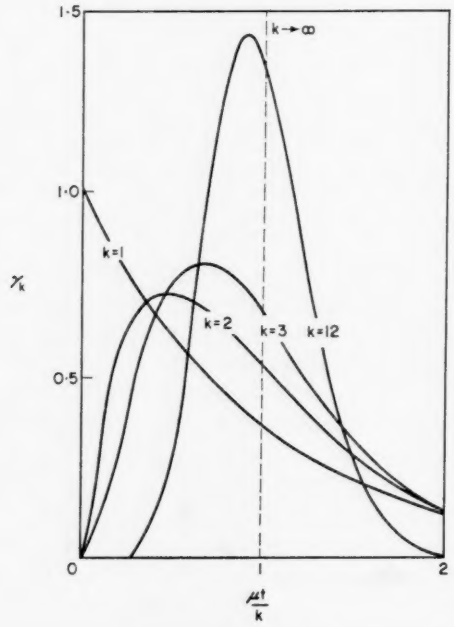

FIG. 1

repair circuit (i.e. provided the components are delivered serviceable to stores in the same order as they are removed from aircraft), it may be seen that the number of unserviceable components just before the end of a "lead time" will equal the number of "demands" during that "lead time". If there is overtaking, then the definition of lead time must be changed by pairing the demands with the deliveries to stores in the order in which they occur so that, despite overtaking, this equality is still preserved. Our second problem may now be tackled exactly as the first, by choosing the float size n so as to give a specified risk P_n (if possible, chosen for the economic optimum exactly as in the first problem) that the number of unserviceable components will exceed n at the end of a lead time. Again we must find P_n as a function of n.

We shall consider the two problems just described in the case where:

(1) The demand is random in the sense that the number of demands r during a fixed interval of time t has a Poisson distribution

$$\frac{e^{-\lambda t}(\lambda t)^r}{\Gamma(r+1)}$$

where λ is the average number of demands per unit time.

(2) The lead time t has a gamma type probability density function

$$\frac{\mu e^{-\mu t}(\mu t)^{k-1}}{\Gamma(k)} \qquad (k>0)$$

with mean k/μ and variance k/μ^2 both independent of n.

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Assumption 1 is often fairly realistic even in the airline problem where a number of demands may be due to scheduled overhauls rather than to component failures.

Assumption 2 is a very general assumption since it covers a range of lead time distributions from constant ($k\to\infty$) to purely random ($k=1$) and includes all intermediate choices of variability. The diagram opposite shows representative gamma type probability density functions (also well known as Pearson Type III distributions).

Under assumptions 1 and 2 we may now calculate the probability p_{rk} of exactly r demands during the lead time when the parameter value is k:

$$p_{rk} = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^r}{\Gamma(r+1)} \times \frac{\mu e^{-\mu t}(\mu t)^{k-1}}{\Gamma(k)}\, dt$$

$$= \frac{\lambda^r \mu^k}{\Gamma(r+1)\,\Gamma(k)} \int_0^\infty t^{r+k-1} e^{-(\lambda+\mu)t}\, dt.$$

Putting $$z = (\lambda+\mu)\,t,$$

this gives

$$p_{rk} = \frac{\lambda^r \mu^k}{(\lambda+\mu)^{r+k}\,\Gamma(r+1)\,\Gamma(k)} \int_0^\infty z^{r+k-1} e^{-z}\, dz$$

$$= \frac{\lambda^r \mu^k}{(\lambda+\mu)^{r+k}} \times \frac{\Gamma(r+k)}{\Gamma(r+1)\,\Gamma(k)}.$$

Confining our interest to integral values of both k and r, this may be rewritten

$$p_{rk} = \binom{r+k-1}{k-1} \frac{(\lambda/\mu)^r}{(1+\lambda/\mu)^{r+k}}. \qquad (1)$$

Thus r has a negative binomial distribution with mean $x = \lambda k/\mu$ and variance $s^2 = \lambda k/\mu\{1+\lambda/\mu\}$. A similar derivation of the negative binomial distribution is given elsewhere.[2]

We shall hereafter refer to a risk level P_{nk} which is related to p_{rk} in (1) by

$$P_{nk} = \sum_{r=n+1}^{\infty} p_{rk}.$$

In the case $k = 1$ we find

$$p_{r1} = \frac{x^r}{(1+x)^{r+1}}, \quad P_{n1} = \left(\frac{x}{1+x}\right)^{n+1},$$

while at the other extreme $k \to \infty$ we find

$$p_{r\infty} = \frac{e^{-x}x^r}{r!}, \quad P_{n\infty} = \sum_{r=n+1}^{\infty} \frac{e^{-x}x^r}{r!}.$$

These two results were used by N. S. Roseman (unpublished work) to produce graphs for BEA giving maximum and minimum float sizes.

Another fairly simple case is $k = 2$, which gives

$$P_{n2} = \left(\frac{x}{x+2}\right)^{n+1}\left[(n+2)-(n+1)\left(\frac{x}{x+2}\right)\right].$$

It is possible to obtain analogous expressions for higher values of k using a general formula for P_{nk} which we shall shortly derive (2); but since such expressions are unwieldy for values of k exceeding 2, and since (as we shall see), it is possible to find numerical values by the use of statistical tables, these expressions are unlikely to be of much interest.

To digress somewhat here, it is interesting to notice that the Poisson solution $P_{n\infty}$ is also obtained from one of the simplest queueing type models of the float calculation problem.[8] This formulation assumes random demand, and an effectively infinite number of parallel service facilities for the repair of components. The difference between this model and the model which gave us P_{n1} is simply the restriction on overtaking which we took into account when defining lead time, and also a slightly different interpretation of "risk level". Bearing in mind the fact that lead time includes both waiting and service, it is not difficult to see that the solution P_{n1} itself results from the corresponding single-server queue with a negative exponential service time distribution.

Returning now to our original model, there are two alternative methods of fitting the distribution of r to actual data (leaving aside the details of distribution fitting). One is separately to fit the lead time distribution and the demand distribution. A more satisfactory approach from the point of view of directness is to fit the negative binomial distribution itself.[1] The method of moments will usually be found adequate for this if $k \geqslant 1$, since the efficiency is at least 50 per cent.[1] A very convenient approach when a number of different types of commodity or component are under consideration, is to plot each type as a point, with the ordinate y as the estimated variance divided by the mean s^2/x and the abscissa as the estimated mean x of the number of demands during the lead time. This will give a scatter of points which will lie close to a straight

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line $y = 1+x/k$ if the types all have the same gamma lead time distribution. If several lead time distributions are involved, then this graphical method may be convenient for separating them.

The stage has been reached where we are able to determine the distribution of the number of demands during a lead time. We now wish to be able to find n, given the risk level P_{nk}.

Now tables of the negative binomial distribution are not easy to find (if, indeed, they exist at all) and it would hence be very useful to find a more convenient way of expressing the risk level. Surprisingly enough, it turns out that the infinite sum of negative binomial terms which gives the risk level, may also be expressed as a finite sum of positive binomial terms and is tabulated as such. A proof is given in the Appendix, the result being

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$$P_{nk} = \sum_{R=0}^{k-1} \binom{N}{R} p^R q^{N-R}, \tag{2}$$

where

$$N = n+k, \quad p = \frac{1}{1+\lambda/\mu} \left(= \frac{x}{s^2} = \frac{k}{x+k} \right)$$

and

$$q = 1-p.$$

Now given n, k and p (or n, x and s^2) one can in general obtain P_{nk} directly from cumulative binomial tables.[7]

If N is too large to use the binomial tables, then for either p or $(1-p)$ small compared with unity, we may use the Poisson approximation giving for small p

$$P_{nk} = \sum_{R=0}^{k-1} \frac{e^{-Np}(Np)^R}{R!}, \tag{3}$$

and for p close to 1

$$P_{nk} = \sum_{S=N-k+1}^{N} \frac{e^{-Nq}(Nq)^S}{S!}, \tag{4}$$

which may be calculated from tables.[6] However, approximations (3) and (4) are not necessarily required even for large N, since a small value of p will often result from a small value of k, while a value of p close to 1 will often result from a very large value of k. Under these circumstances, one of the limiting cases, $k = 1, 2, \infty$, which we have already dealt with, may be used.

If N is large and p lies between say 0·2 and 0·8, then the normal approximation may be used, giving

$$P_{nk} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{z} e^{-\frac{1}{2}z^2} dz \quad \text{where} \quad z = \frac{(k-1)-Np}{\sqrt{(Npq)}}. \tag{5}$$

Let us sum up the procedures necessary to fix the reorder level or float size n of a commodity or component. One first finds the parameters of the negative

binomial distribution of number of demands during the lead time. If there is a satisfactory fit, one now employs (2) or the appropriate approximation, in conjunction with tables, to give P_{nk} for a range of possible values of n. Finally, one chooses that n which corresponds to a specified value of P_{nk}. A normal approximation to the negative binomial distribution may be used (as in the following example) either to check or to obtain a first approximation to the value of n, provided n and k are not too small.

As a numerical example, let us consider a case where the average number of demands during the lead time has been estimated as $x = 30$ and the variance of number of demands during the lead time has been estimated as $s^2 = 75$. Fitting the negative binomial distribution by the method of moments, we find

$$k = \frac{x^2}{s^2 - x} = 20 \quad \text{and} \quad p = \frac{x}{s^2} = 0{\cdot}4.$$

Obviously N will be greater than $x+k = 50$ and hence we must resort to approximation (5).

Let us suppose the acceptable risk level is 0·05. Normal tables[4] give us $z = -1{\cdot}645$. Hence

$$z^2 = (1{\cdot}645)^2 = \frac{[19 - 0{\cdot}4(n+20)]^2}{(n+20) \times 0{\cdot}4 \times 0{\cdot}6}.$$

This reduces to the quadratic equation

$$0{\cdot}16n^2 - 9{\cdot}45n + 108{\cdot}0 = 0.$$

The larger of the two roots (43·6) gives the only practicable solution which, rounded up to the nearest integer, produces the result

$$n = 44.$$

The crude normal approximation gives $n = 30 + 1{\cdot}645\sqrt{(75)} = 44{\cdot}2$, which rounds down to the same result in this example.

ACKNOWLEDGEMENT

Acknowledgement is made to British European Airways for permission to publish this paper, and to Mr. A. M. Lee for his helpful suggestions on its presentation.

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APPENDIX

To prove that

$$P_{nk} = \sum_{R=0}^{k-1} \binom{N}{R} p^R q^{N-R},$$

where

$$N = n+k, \quad p = \frac{1}{1+\lambda/\mu} \quad \text{and} \quad q = 1-p,$$

we have by definition

$$P_{nk} = \sum_{r=n+1}^{\infty} p_{rk},$$

where the p_{rk} are individual negative binomial terms.

Writing $t = r-n$, and substituting for p_{rk} the expression (1), we obtain

$$P_{nk} = \sum_{t=1}^{\infty} \binom{n+t+k-1}{k-1} p^k q^{n+t}. \qquad \text{(A)}$$

Multiplying each term on the right-hand side of (A) by unity expressed as $(1-q)/p$, this gives

$$P_{nk} = \sum_{t=1}^{\infty} \left\{ \binom{n+t+k-1}{k-1} p^{k-1} q^{n+t} - \binom{n+t+k-1}{k-1} p^{k-1} q^{n+t+1} \right\}.$$

Collecting like terms, we find the coefficient of $p^{k-1}q^{n+1}$ is

$$\binom{n+k}{k-1},$$

while for $t>0$ the coefficient of $p^{k-1}q^{n+t+1}$ is

$$\binom{n+t+k}{k-1} - \binom{n+t+k-1}{k-1} = \binom{n+t+k-1}{k-2}.$$

Hence

$$P_{nk} = \binom{n+k}{k-1} p^{k-1} q^{n+1} + \sum_{t=1}^{\infty} \binom{n+t+k-1}{k-2} p^{k-1} q^{n+t+1}$$

$$= \binom{n+k}{k-1} p^{k-1} q^{n+1} + P_{n+1,k-1}.$$

Replacing n by $n+t-1$ and k by $k-t+1$, this recurrence relation becomes

$$P_{n+t-1,k-t+1}-P_{n+t,k-t}=\binom{n+k}{k-t}p^{k-t}q^{n+t}.$$

Summing both sides from $t=1$ to $k-1$, this gives

$$P_{n,k}-P_{n+k-1,1}=\sum_{t=1}^{k-1}\binom{n+k}{k-t}p^{k-t}q^{n+t}.$$

Now by (A),

$$P_{n+k-1,1}=\sum_{t=1}^{\infty}\binom{n+t+k-1}{0}pq^{n+t+k-1}$$
$$=q^{n+k}.$$

Hence

$$P_{nk}=\sum_{t=1}^{k}\binom{n+k}{k-t}p^{k-t}q^{n+t},$$

and writing $N=n+k$, $R=k-t$, this proves the result.

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On Analyses of the Solution to a Linear Programming Problem†

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Many complex problem situations in various contexts have been represented in recent years by the *linear programming model*. The *simplex method* can then be used to give the optimal values of the variables corresponding to a given set of values of the parameters. However, in many situations it is useful to have the solution to many other related problems which differ from the original problem only in the values of some of the parameters. This paper presents procedures by which the solutions to the changed problems can be derived from the simplex solution tableau corresponding to the original problem. The method will be illustrated by means of an example problem, and it will be shown how quantitative information obtained from such analyses can aid management in decision making.

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THE simplex method formulated by Danzig[6] provides a convenient and systematic method of analysing linear programming problems. A linear programming model can be expressed as: Find non-negative values of variables x_j $(j = 1, \ldots, n)$, so as to maximize

$$\sum c_j x_j, \tag{1}$$

satisfying the conditions

$$\sum_{j \notin \mathbf{B}} a_{ij} x_j + x_i = a_{i0} \quad (i \in \mathbf{B}), \tag{2}$$

where x_i $(i \in \mathbf{B})$ form the basic variables in the initial simplex tableau. The model assumes that the values of the parameters a_{ij} and c_j are fixed and known. However, in many cases, it is useful to view the problem under less restrictive assumptions. For example, Saaty[11,12] has investigated the case where the parameters can be expressed in parametric form. On the other hand, Heller[7] has studied the effect on the criterion function due to an incremental change in the value of a parameter. An interesting aspect of the problem is when certain parameters can take on an arbitrary set of new values. This paper discusses procedures for obtaining the solution to such a changed problem, starting with the simplex solution tableau for the original problem. Such analyses have found useful practical applications. For example, Charnes, Cooper and Mellon[4,5] have discussed it with reference to oil refinery operations, and the study

† Based mainly on part of the study presented in a Ph.D. dissertation [14].

‡ The author wishes to express his appreciation to the Editor, *Journal of Industrial Engineering*, for the kind permission to include in this paper results published in [13]. The author is also indebted to Dr. A. Charnes for pioneering work in the area covered by this paper and to Dr. L. G. Mitten for his comments on the manuscript.

presented in reference 3 shows its applications in a manufacturing operation. Some of the other investigations into the methodology and applications of such studies may be found in references 1, 2, 8, 9, 13, 14.

For the sake of convenience, we will first introduce the notation used in this presentation and will state a numerical example problem. The solution procedures will then be presented under two major sections, namely,

(1) change in the values of two or more parameters,

(2) change in the value of a single parameter.

Further, for the sake of clarity, we will illustrate the procedures by means of numerical examples. We will also present meaningful interpretations of the results of the analyses in the context of the example problem.

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NOTATION

Consider the initial tableau (e.g. Table 2, given later). Corresponding to this tableau we have already defined an index set $\mathbf{B} = \{j \mid x_j \text{ is a basic variable in the initial tableau}\}$. We will denote by a_{IJ} the element at the intersection of a row corresponding to a variable x_I ($I \in \mathbf{B}$) and the column corresponding to a variable x_J. Likewise, the element a_{I0} will be in x_I's row. We will denote by $\mathbf{a}_j$ the $m \times 1$ column matrix containing the elements a_{ij} ($j = 0, \ldots, n$). The elements in the last two rows of the tableau will then represent

$$z_j = \sum_{i \in \mathbf{B}} c_i a_{ij} \quad \text{and} \quad w_j = (c_j - z_j) \quad (j = 1, \ldots, n).$$

Furthermore, to distinguish between the elements of the initial tableau and those of the final tableau (Tables 2 and 3 respectively, given later), we will denote the latter with a prime. To illustrate the notation, in Tables 2 and 3 we have: $\mathbf{B} = \{5, 6, 7\}$, $a_{52} = 0{\cdot}6$, $a_{60} = 180$, $\mathbf{B}' = \{1, 3, 2\}$, $z'_4 = 20$, $w'_4 = -4$ and a'_6 is the column matrix containing the elements $(-2{\cdot}4,\ 4{\cdot}0,\ 0{\cdot}8)$.

Now consider the columns $j \in \mathbf{B}'$ in the final tableau. Rearrange them to form an identity matrix. Let the corresponding columns $j \in \mathbf{B}'$ in the initial tableau also simultaneously be rearranged in the same order. We will denote the $m \times m$ matrix thus obtained by $\mathbf{T}$. Next consider the columns $j \in \mathbf{B}$ in the initial tableau, and rearrange them (if necessary) to form an identity matrix. It can be shown that the $m \times m$ matrix formed by rearranging the corresponding columns $j \in \mathbf{B}$ of the final tableau in the same order, will give $\mathbf{T}^{-1}$, the inverse matrix of $\mathbf{T}$.

Thus, in the illustrative problem given later, we have

$$\mathbf{T} = \begin{array}{c} \begin{matrix} x_1 & x_3 & x_2 \end{matrix} \\ \begin{bmatrix} 0{\cdot}2 & 0 & 0{\cdot}6 \\ 0 & 0{\cdot}25 & 0 \\ -1{\cdot}0 & -1{\cdot}0 & 2{\cdot}0 \end{bmatrix} \begin{matrix} x_5 \\ x_6 \\ x_7 \end{matrix} \end{array} \quad \text{and} \quad \mathbf{T}^{-1} = \begin{array}{c} \begin{matrix} x_5 & x_6 & x_7 \end{matrix} \\ \begin{bmatrix} 2{\cdot}0 & -2{\cdot}4 & -0{\cdot}6 \\ 0 & 4{\cdot}0 & 0 \\ 1{\cdot}0 & 0{\cdot}8 & 0{\cdot}2 \end{bmatrix} \begin{matrix} x_1 \\ x_3 \\ x_2 \end{matrix} \end{array}.$$

In the foregoing matrices, the variables corresponding to the columns and rows are indicated for convenience. This will facilitate relating the elements of the matrix with the equations given later under Algorithms 1 through 3. It may also be noted that the definitions of the elements a_{ij} and a'_{ij} given earlier will continue to apply in the case of matrices $\mathbf{T}$ and $\mathbf{T}^{-1}$ respectively. Furthermore, the columns corresponding to a variable x_j in matrix $\mathbf{T}$ will be the same as in the initial tableau (which we have already denoted by $\mathbf{a}_j$). Similarly, the columns of $\mathbf{T}^{-1}$ will correspond to the columns of the final tableau denoted by $\mathbf{a}'_j$. In the algorithms given later, we will also have occasion to refer to certain rows of $\mathbf{T}$ and $\mathbf{T}^{-1}$. We will denote the row corresponding to a variable x_i in $\mathbf{T}$ by $\vec{\mathbf{a}}_i$ $(i \in \mathbf{B})$. For example, in the case of the matrix $\mathbf{T}$ given above, we have $\vec{\mathbf{a}}_5 = (0{\cdot}2, 0, 0{\cdot}6)$. Likewise, we will denote the row corresponding to a variable x_i in $\mathbf{T}^{-1}$ by $\vec{\mathbf{a}}'_i$ $(i \in \mathbf{B}')$.

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Now consider a change in the value of a parameter. For example, let a_{ij} be changed to a^*_{ij}. We will denote the change by $\Delta a_{ij} = (a^*_{ij} - a_{ij})$. When the elements in row I of the initial tableau are changed, the elements of $\vec{\mathbf{a}}_I$ in $\mathbf{T}$ will also change in general. The $1 \times m$ matrix of changes in row I of $\mathbf{T}$, i.e. $[\Delta a_{Ij}]$ $(j \in \mathbf{B}')$, will be denoted by $\Delta\vec{\mathbf{a}}_I$. Likewise, when the elements of column J are changed, we have the $m \times 1$ matrix of changes in $\mathbf{a}_J$ denoted by $\Delta\mathbf{a}_J = [\Delta a_{iJ}]$ $(i \in \mathbf{B})$.

ILLUSTRATIVE PROBLEM

For the sake of clarity, the algorithms given later for the analyses of the problem will be illustrated using the following hypothetical example. Hamilton Manufacturing Company Ltd. desires to manufacture two components, *A* and *B*, of an assembly. An assembly uses two pieces of *A* and one piece of *B*, and these components are to be manufactured in this ratio. Each of these components can be manufactured with either of the two machines *I* and *II*. Table 1 below gives the time in hours required and the associated profit per piece for each component manufactured on a particular machine. Let us assume the machine times available on machines *I* and *II* are 240 and 180 hours, respectively. Furthermore, suppose the idle time on machine *I* will incur a loss of \$10.00 per hour.

TABLE 1

	Machine availability (hrs)	Production rate (hrs/piece)		Profit per piece (\$)	
		A	*B*	*A*	*B*
Machine *I*	240	0·2	0·6	10	20
Machine *II*	180	0·25	1·0	8	16

To express the problem mathematically, let x_1 and x_2 denote the number of pieces of components *A* and *B*, respectively, manufactured on machine *I*, and let x_3 and x_4 represent the same for machine *II*. Setting up the time-balance equations for the two machines, we have

$$0{\cdot}2x_1+0{\cdot}6x_2 \qquad +x_5 \qquad = 240, \tag{3}$$

$$0{\cdot}25x_3+x_4 \qquad +x_6 = 180, \tag{4}$$

where x_5 and x_6 are the slack variables representing the idle time in hours on machines *I* and *II*, respectively. Further, we must have

$$(x_1+x_3) = 2(x_2+x_4), \tag{5}$$

which can be written as

$$-x_1+2x_2-x_3+2x_4+x_7 = 0, \tag{5a}$$

where x_7 is a dummy variable (equal to zero in the solution to the problem). The function to be maximized is then given by

$$C(\mathbf{x}) = 10x_1+20x_2+8x_3+16x_4-10x_5-Mx_7, \tag{6}$$

where M is a large positive number introduced to ensure that the variable x_7 is equal to zero in the solution so that condition (5) is satisfied.

We can now form the initial simplex tableau as shown at Table 2. Proceeding by the simplex method, we have the final solution tableau given at Table 3.

TABLE 2

		c_j	10	20	8	16	−10	0	$-M$
c_i	Basis	$\mathbf{a}_0$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
−10	x_5	240	0·2	0·6	0	0	1	0	0
0	x_6	180	0	0	0·25	1·0	0	1	0
$-M$	x_7	0	−1·0	2·0	−1·0	2·0	0	0	1
	z_j		$-2+M$	$-6-2M$	M	$-2M$	−10	0	$-M$
$w_j=(c_j-z_j)$			$12-M$	$26+2M$	$8-M$	$16+2M$	0	0	0

TABLE 3

		c_j	10	20	8	16	−10	0	$-M$
c_i	Basis	$\mathbf{a}_0'$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
10	x_1	48	1	0	0	−3·6	2·0	−2·4	−0·6
8	x_3	720	0	0	1	4·0	0	4·0	0
20	x_2	384	0	1	0	1·2	1·0	0·8	0·2
	z_j	13,920	10	20	8	20	40	24	−2
$w_j'=(c_j-z_j')$			0	0	0	−4	−50	−24	$-M+2$

CHANGES IN TWO OR MORE VARIABLES

Before presenting the algorithm for solving the general problem with arbitrary changes in the values of two or more parameters, we will discuss the solution procedures for two special cases. These deal with situations where the elements of the matrix $\mathbf{A} = [a_{ij}]$ $(i = 1, \ldots, m; j = 1, \ldots, n)$ remain unchanged except for:

Case 1

Changes in elements in a certain row, e.g. when an existing restriction is replaced by another. As an example of such a situation, suppose a certain raw material (input) is to be replaced by another (say, of a better quality) so that the different *activities* (outputs) will consume the new raw material at rates different from that in the case of the original raw material. Thus the restriction expressing the utilization of the raw material has to be modified.

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Case 2

Changes in elements in a certain column, e.g. when the resources consumed by an activity changes. Such is the situation when a new product is to be introduced, replacing another product being manufactured. The new product can be expected to require the different resources at rates different from that needed by the original product.

Algorithms 1 and 2 that follow deal with the above cases. Algorithm 3, given later, deals with the general case involving arbitrary changes in the values of the parameters and uses Algorithms 1 and 2 as subroutines in the solution procedure. Following the presentation of the algorithms, an illustration of their use will be given.

CHANGE IN ROW I OF MATRIX A

i.e. with $\Delta a_{ij} = 0$, $i \neq I$, $j = 1, \ldots, n$

We now proceed to give the algorithm for the case where all values of a_{ij} ($i \in \mathbf{B}$ and $j = 1, \ldots, n$) are unchanged except, perhaps, in row I.

Algorithm 1

Step 1. Consider the current inverse matrix, $\mathbf{T}^{-1} = [a'_{ij}]$ $(i \in \mathbf{B}', j \in \mathbf{B})$. Change its elements by

$$\Delta a'_{ij} = -\frac{a'_{iI} \sum_{k \in \mathbf{B}'} \Delta a_{Ik} a'_{kj}}{1 + \sum_{k \in \mathbf{B}'} \Delta a_{Ik} a'_{kI}} = -\frac{a'_{iI} \Delta \vec{\mathbf{a}}_I \mathbf{a}'_j}{\alpha_I},$$

where $\Delta \vec{\mathbf{a}}_I$ is a $1 \times m$ matrix $= [\Delta \mathbf{a}_{Ij}]$ $(j \in \mathbf{B}')$ and $\alpha_I = (1 + \Delta \vec{\mathbf{a}}_I \mathbf{a}'_I)$. Denote the matrix thus obtained by $\mathbf{T}^{*-1}$, and proceed to Step 2.

Step 2. Form a new tableau whose jth column $\mathbf{a}_j$ $(j = 0, \ldots, n)$ is given by

$$\mathbf{a}_j = \begin{cases} \mathbf{a}'_j & (j \in \mathbf{B}') \\ \mathbf{T}^{*-1} \mathbf{a}^*_j & (j = 0, j \notin \mathbf{B}') \end{cases}$$

where $\mathbf{a}_j^*$ is the $m \times 1$ matrix with the changed elements a_{ij}^* $(i \in \mathbf{B})$. If all $\mathbf{a}_{i0} \geqq 0$ $(i \in \mathbf{B}')$, proceed to Step 3. Otherwise, proceed to Step 4.

Step 3. Enter the values of c_j^* in the appropriate column and row. Compute

$$\mathbf{z}_j = \sum_{i \in \mathbf{B}'} c_i^* \mathbf{a}_{ij} \quad \text{and} \quad \mathbf{w}_j = (c_j^* - \mathbf{z}_j) \quad (j = 1, \ldots, n).$$

If we have $\mathbf{w}_j \leqq 0$ $(j \notin \mathbf{B}')$, the tableau is the solution tableau. Otherwise, apply the simplex method to obtain the solution tableau (with a different basis).

Step 4. Let $\mathbf{N} = \{i \,|\, \mathbf{a}_{i0} < 0 \; (i \in \mathbf{B}')\}$. Let there be h elements in this set. Form an augmented tableau as follows:

(a) Replace the variables x_i $(i \in \mathbf{N})$ (in the column titled "basis") by dummy variables $x_{n+1}, \ldots, x_{n+h}$. Change the sign of the elements a_{ij} $(j = 0, \ldots, n)$ in these rows.

(b) Add h new columns to the tableau corresponding to the h dummy variables added with zero elements in all rows except in the row in which it is the basis variable. In the latter, the value is 1. Proceed to Step 3 with $c_j^* = -M$ $(j = n+1, \ldots, n+h)$, where M is a large positive number.

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Changes in Column J of Matrix $\mathbf{A}$

i.e. with $\Delta a_{ij} = 0 \quad (i \in \mathbf{B}, j \neq J)$

We now consider the problem where all values of a_{ij} $(i \in \mathbf{B}$ and $j = 1, \ldots, n)$ are unchanged except, perhaps, in column J. This case can be handled by two alternative methods and presented below. The first procedure is analogous to Algorithm 1 given earlier.

Algorithm 2a

Step 1. Consider the inverse of the current basis, i.e. $\mathbf{T}^{-1} = [a'_{ij}]$ $(i \in \mathbf{B}', j \in \mathbf{B})$. Change its elements by†

$$\Delta a'_{ij} = -\frac{a'_{Jj} \sum_{k \in \mathbf{B}} a'_{ik} \Delta a_{kJ}}{1 + \sum_{k \in \mathbf{B}} a'_{Jk} \Delta a_{kJ}} = -\frac{a'_{Jj} \vec{\mathbf{a}}'_i \Delta \mathbf{a}_J}{\beta_J},$$

where $\vec{\mathbf{a}}'_i$ is a $1 \times m$ matrix $= [a'_{ik}]$ $(k \in \mathbf{B})$ and $\beta_J = (1 + \vec{\mathbf{a}}'_J \Delta \mathbf{a}_J)$. Denote the matrix thus obtained by $\mathbf{T}^{*-1}$, and proceed to Step 2 of Algorithm 1.

An alternative procedure for the solution of the above problem can be derived by recalling that the simplex method proceeds from one tableau to another precisely by replacing one column vector by another in the current basis. The simplex method can, therefore, be applied in the case under consideration with obvious minor modifications as indicated below.

† $\Delta a'_{ij} = 0$ if we have $J \notin \mathbf{B}'$. In this case proceed directly to Step 2 of Algorithm 1 with $\mathbf{T}^{-1} = \mathbf{T}^{*-1}$.

Algorithm 2b

Step 1. Consider the final tableau. If $J \notin \mathbf{B}'$, proceed directly to Step 2. Otherwise, change the variable x_J to $x_{J'}$ (both in the column titled "basis" and in the row indicating the variables). Change its coefficient to $-M$ where M is a large positive number, i.e. $c_{J'} = -M$. Add a column for the variable x_J, and enter the corresponding coefficient c_J. Proceed to Step 2.

Step 2. Let $\mathbf{a}_J^*$ be the changed $m \times 1$ matrix in the initial tableau. Premultiply $\mathbf{a}_J^*$ by $\mathbf{T}^{-1}$, and enter the values in the column corresponding to variable x_J. Proceed to Step 3.

Step 3. Apply the simplex algorithm until the solution is reached.

ARBITRARY CHANGES IN THE VALUES OF THE PARAMETERS

As in the cases discussed above, arbitrary simultaneous changes in the values of the parameters can be studied by first considering the changes in the elements of the matrix **T**. This can be done by selecting certain rows and/or columns of **T** such that they contain all the changed elements in **T**. Repeated application of Algorithms 1 and 2 will then lead to the optimal solution. The details of the procedure are as follows.

Algorithm 3

Step 1. Consider the matrix $\mathbf{T} = [a_{ij}]$ ($i \in \mathbf{B}'$ and $j \in \mathbf{B}'$). Find a set of rows and/or columns such that all the changed elements in **T** are contained in them.† Let **R** denote the set of subscripts of the variables corresponding to the rows selected and let **C** denote the corresponding set for the columns. For every $a_{ij}^* \neq a_{ij}$ we then have either $i \in \mathbf{R}$ and/or $j \in \mathbf{C}$. Proceed to Step 2.

Step 2. Select a (previously unselected) element r of **R**. Consider the matrix $\bar{\mathbf{T}}^{-1} = [\bar{a}_{ij}]$.‡ Change its elements by

$$\Delta \bar{a}_{ij} = -\frac{\bar{a}_{ir} \sum_{k \in \mathbf{B}'} \Delta a_{rk} \bar{a}_{kj}}{1 + \sum_{k \in \mathbf{B}'} \Delta a_{rk} \bar{a}_{kr}} = -\frac{\bar{a}_{ir} \Delta \vec{\mathbf{a}}_r \bar{\mathbf{a}}_j}{\alpha_r} \quad (i \in \mathbf{B}', j \in \mathbf{B}),$$

where $\alpha_r = (1 + \Delta \vec{\mathbf{a}}_r \bar{\mathbf{a}}_r)$ and $\Delta \vec{\mathbf{a}}_r$ is a $m \times 1$ matrix $= [\Delta a_{rj}]$ ($j \in \mathbf{B}'$). Denote the elements of the matrix thus obtained by $\bar{a}_{ij}$, and repeat Step 2 of this algorithm for every element of **R**. When all the elements of the set **R** have been considered, proceed to Step 3.

† The computations involved in subsequent steps will considerably be reduced if we can find the smallest set of rows and columns containing all the elements $a_{ij}, j \in \mathbf{B}'$, which are changed. In other words, if we call these the "*zero*" *elements*, and the rows and columns of the matrix as *lines*, we wish to find the *minimal set of lines* containing all the "zero" elements. An efficient procedure for this purpose is given in 10 under the section on the assignment problem.

‡ For the first iteration $\bar{\mathbf{T}}^{-1} = \mathbf{T}^{-1}$.

Step 3. Select a (previously unselected) element c of **C**. Consider the matrix $\bar{\mathbf{T}}^{-1} = [\bar{a}_{ij}]$ obtained at the end of Step 2 above. Change its elements by

$$\Delta\bar{a}_{ij} = -\frac{\bar{a}_{cj}\sum_{k\in\mathbf{B}}\bar{a}_{ik}\Delta a_{kc}}{1+\sum_{k\in\mathbf{B}}\bar{a}_{ck}\Delta a_{kc}} = -\frac{\bar{a}_{cj}\vec{\bar{\mathbf{a}}}_i\Delta\mathbf{a}_c}{\beta_c} \quad (i\in\mathbf{B}', j\in\mathbf{B}),$$

where $\beta_c = (1+\vec{\bar{\mathbf{a}}}_c\Delta\mathbf{a}_c)$, $\vec{\bar{\mathbf{a}}}_i$ is a $1\times m$ matrix $= [\bar{a}_{ik}]$, and $\Delta\mathbf{a}_c$ is an $m\times 1$ matrix with elements

$$\Delta a_{ic} = \begin{cases} 0 & \text{for } i\in\mathbf{R}, \\ (a^*_{ic}-a_{ic}) & \text{for } i\notin\mathbf{R}. \end{cases}$$

Repeat Step 3 of this algorithm for every element of **C**. Denote the matrix thus obtained by $\mathbf{T}^{*-1}$ and proceed to Step 2 of Algorithm 1.

NUMERICAL EXAMPLE

We will now use the example problem given earlier to illustrate Algorithms 1 through 3, given above. Suppose machine *I* is to be replaced by another (call it *I'*) available for 270 hours and with production rates of 0·1 and 0·4 hours per piece for components *A* and *B*, respectively. Moreover, suppose a methods improvement for production on machine *II* permits a decrease in the time required for production to 0·18 and 0·9 hours per piece, respectively, for the two components. Taking the above factors into consideration, assume that the profits per piece are as given in Table 4 below, which also summarizes the other

TABLE 4

	Machine availability (hrs.)	Production rate (hrs./piece)		Profit per piece ($)	
		A	*B*	*A*	*B*
Machine *I*	270	0·1	0·4	15	24
Machine *II*	180	0·18	0·9	13	19

data mentioned above. Furthermore, suppose that 10 per cent of components *A* manufactured on machine *I* turn out to be defective and have to be sold at a reduced price of $3.00 per piece.

If we use the notation x_1 and x_2 to represent the amounts of components *A* and *B*, respectively, manufactured on machine *I'*, the new initial tableau will be as shown in Table 5 opposite. It may be noted that we have $a^*_{73} = -0{\cdot}9$ as only 90 per cent of components *A* manufactured on machine *II* are usable. For the same reason, the average profit for a component *A* manufactured on machine *II* is given by $c^*_3 = 13(0{\cdot}9)+3(0{\cdot}1) = 12$. The other entries in the table are obvious.

We will now solve the problem using Algorithm 3. At Step 1 we let $\mathbf{R} = \{5\}$ and $\mathbf{C} = \{3\}$. We now apply Step 2 for changes in row $r = 5$. We have

$$\Delta\vec{\mathbf{a}}_5 = (-0{\cdot}1, 0, -0{\cdot}2) \quad \text{and} \quad \alpha_5 = 1 + (-0{\cdot}1, 0, -0{\cdot}2)(2{\cdot}0, 0, 1{\cdot}0)^t = 0{\cdot}6$$

where the superscript t denotes the transpose of the column matrix. We can now evaluate $\Delta\bar{a}_{ij}$.

TABLE 5

		c_j^*	15	24	12	19	−10	0	$-M$
c_i^*	Basis	$\mathbf{a}_0^*$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
−10	x_5	270	0·1	0·4	0	0	1	0	0
0	x_6	180	0	0	0·18	0·9	0	1	0
$-M$	x_7	0	−1·0	2·0	−0·9	2·0	0	0	1

For the sake of brevity, we will give below the complete computations for evaluating $\Delta\bar{a}_{15}$ and will present only the results of other evaluations. We have

$$\Delta\bar{a}_{15} = -\frac{2(-0{\cdot}1, 0, -0{\cdot}2)(2{\cdot}0, 0, 1{\cdot}0)^t}{0{\cdot}6} = -\frac{2(-0{\cdot}4)}{0{\cdot}6} = 4/3,$$

and the changes needed in all the elements are given by

$$[\Delta\bar{a}_{ij}] = \begin{bmatrix} 4/3 & 4/15 & -1/15 \\ 0 & 0 & 0 \\ 2/3 & 2/15 & -1/30 \end{bmatrix},$$

giving the following new inverse matrix:

$$\bar{\mathbf{T}}^{-1} = \begin{array}{c} \begin{matrix} x_5 & x_6 & x_7 \end{matrix} \\ \begin{bmatrix} 10/3 & -8/3 & -2/3 \\ 0 & 4 & 0 \\ 5/3 & 2/3 & 1/6 \end{bmatrix} \end{array} \begin{matrix} x_1, \\ x_3, \\ x_2. \end{matrix}$$

We now proceed to Step 3 of Algorithm 3. We have, for $c = 3$,

$$\Delta\mathbf{a}_3 = (0, -0{\cdot}07, 0{\cdot}1)^t \quad \text{and} \quad \beta_3 = 1 + (0, 4{\cdot}0, 0)(0, -0{\cdot}07, 0{\cdot}1)^t = 0{\cdot}72.$$

The matrix of changes is then readily found to be

$$[\Delta\bar{a}_{ij}] = \begin{bmatrix} 0 & -2/3 & 0 \\ 0 & 14/9 & 0 \\ 0 & 1/6 & 0 \end{bmatrix},$$

from which we have

$$\bar{\mathbf{T}}^{*-1} = \begin{bmatrix} 10/3 & -10/3 & -2/3 \\ 0 & 50/9 & 0 \\ 5/3 & 5/6 & 1/6 \end{bmatrix}.$$

Application of Steps 2 and 3 of Algorithm 1 gives us the solution tableau shown in Table 6.

TABLE 6

		c_j^*	15	24	12	19	−10	0	$-M$
c_i^*	Basis	$\mathbf{a}_0'$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
15	x_1	300	1	0	0	−13/3	10/3	−10/3	−2/3
12	x_3	1,000	0	0	1	5	0	50/9	0
24	x_2	600	0	1	0	13/12	5/3	5/6	1/6
		$\mathbf{z}_j'$	30,900	15	24	12	21	80	110/3	−6
$\mathbf{w}_j' = (c_j^* - \mathbf{z}_j')$			0	0	0	−2	−90	−110/3	$-M+6$

CHANGE IN A SINGLE PARAMETER

We have thus far discussed the general problem where the values of an arbitrary set of parameters are changed. We will now consider a special case where the value of one parameter is changed, since such a study provides the means for a critical examination of the solution to a linear programming problem and can result in a deeper understanding of its limitations. If the solution to the changed problem has the same *basis* as the solution to the original problem, the new solution can be derived with almost trivial computations. The algorithms that follow give the procedure for the purpose. In a subsequent section, we will illustrate the procedure by means of numerical examples and will give examples on how meaningful practical interpretations can be given to the information obtained from such analyses.

Before presenting the algorithms, however, we will introduce some additional notation. Consider a change in the value of a parameter. For example, let a_{ij} be changed to a_{ij}^*. We have denoted the change by $\Delta a_{ij} = (a_{ij}^* - a_{ij})$. Let the solution to the changed problem be $x_j^* = (x_j + \Delta x_j)$ with the corresponding value of the criterion function, $C(\mathbf{x}^*) = C(\mathbf{x}) + \Delta C(\mathbf{x})$. In Algorithms 4 through 6 that follow, we will restrict the discussion to the situation where $x_j^* = 0$ $(j \notin \mathbf{B}')$, i.e. when the new solution does not necessitate a change of basis. We will denote by $\bar{\Delta} a_{ij}$ and $\underline{\Delta} a_{ij}$, respectively, the maximum and minimum possible values of Δa_{ij} which will still result in a solution tableau with the basis represented by $\mathbf{B}'$.

Change in c_j

In this section we will study the effect of changing c_j $(j = 1, \ldots, n)$, the coefficients of the variables in the criterion function.

Algorithm 4a

Change in c_J $(J \notin \mathbf{B}')$, i.e. x_J is not in the basis in the final tableau.

(i) The maximum possible change in c_J without necessitating a change of basis (so that we still have $x_j^* = 0$ $(j \notin \mathbf{B}')$) is given by

$$\underline{\Delta} c_J \leqq \Delta c_J \leqq \overline{\Delta} c_J \quad (J \notin \mathbf{B}'),$$

where $\underline{\Delta} c_J = -\infty$ and $\overline{\Delta} c_J = -w'_J$.

(ii) For a change in c_J within the limits indicated above, the optimality of the solution is not affected, i.e. $\Delta x_j = 0$ $(j = 1, \ldots, n)$.

(iii) For a change in c_J within the limits indicated above, the value of the criterion function is unaffected, i.e. $\Delta C(\mathbf{x}) = 0$.

Algorithm 4b

Change in c_J $(J \in \mathbf{B}')$, i.e. x_J is a basic variable in the final tableau.

(i) The maximum change possible in c_J without changing the basis in the final tableau is given by

$$\underline{\Delta} c_J = \Delta c_J = \overline{\Delta} c_J,$$

where

$$\underline{\Delta} c_J = \max \left[\frac{w'_k}{a'_{Jk}}, -\infty \right], \quad a'_{Jk} > 0 \quad \text{and} \quad k \notin \mathbf{B}',$$

$$\overline{\Delta} c_J = \min \left[\frac{w'_k}{a'_{Jk}}, \infty \right], \quad a'_{Jk} < 0 \quad \text{and} \quad k \notin \mathbf{B}'.$$

(ii) For a change in c_J within the limits indicated above, the optimality of the solution is not affected, i.e. $\Delta x_j = 0$ $(j = 1, \ldots, n)$.

(iii) For a change in c_J within the limits indicated above, the change in the value of the criterion function is given by

$$\Delta C(\mathbf{x}) = \Delta c_J \, a'_{J0}.$$

Change in a_{i0}

We now proceed to study the effect on the solution when the value of a restraining constant a_{i0} $(i \in \mathbf{B})$ is changed.

Algorithm 5

(i) The solution to the problem with a parameter a_{I0} changed by Δa_{I0} will have the same basic variables as the original problem if we have

$$\underline{\Delta} a_{I0} \leqq \Delta a_{I0} \leqq \overline{\Delta} a_{I0}$$

where

$$\underline{\Delta}a_{I0} = \max\left[\frac{-a'_{j0}}{a'_{jI}}, -\infty\right], \quad a'_{jI} > 0 \quad \text{and} \quad j \in \mathbf{B}'$$

and

$$\overline{\Delta}a_{I0} = \min\left[\frac{-a'_{j0}}{a'_{jI}}, \infty\right], \quad a'_{jI} < 0 \quad \text{and} \quad j \in \mathbf{B}'.$$

(ii) For a change in a_{I0} within the limits indicated above, the changes necessary in the values of the variables to give an optimal solution to the changed problem are given by

$$\Delta x_j = \begin{cases} \Delta a_{I0}\, a'_{jI} & (j \in \mathbf{B}') \\ 0 & (j \notin \mathbf{B}'). \end{cases}$$

(iii) The change in the value of the criterion function corresponding to the changes in the values of the variables as indicated above is given by

$$\Delta C(\mathbf{x}) = \Delta a_{I0}\, z'_I.$$

Change in a_{ij}

We now consider the effect on the solution when one of the coefficients a_{ij} ($i \in \mathbf{B}$ and $j = 1, \ldots, n$) in the restraining equations is changed. When a particular coefficient a_{IJ} is changed, the limits on the amount of change to ensure that we still have an optimal solution with basis $\mathbf{B}'$ is given by

$$\underline{\Delta}a_{IJ} \leqq \Delta a_{IJ} \leqq \overline{\Delta}a_{IJ}$$

where the values of $\underline{\Delta}a_{IJ}$ and $\overline{\Delta}a_{IJ}$ are as given under (i) in each of the algorithms that follow. The changes necessary in the values of the variables to make them optimal and the corresponding change in the value of the criterion function are given under (ii) and (iii), respectively.

Algorithm 6a

Change in a_{IJ} ($J \in \mathbf{B}'$), i.e. x_J is not a basic variable in the final tableau.

(i) The values of $\underline{\Delta}a_{IJ}$ and $\overline{\Delta}a_{IJ}$ are given by

$$\underline{\Delta}a_{IJ} = \begin{cases} -\infty & \text{when} \quad z'_I \leqq 0 \\ \dfrac{w'_J}{z'_I} & \text{when} \quad z'_I > 0 \end{cases}$$

and

$$\overline{\Delta}a_{IJ} = \begin{cases} \infty & \text{when} \quad z'_I \geqq 0 \\ \dfrac{w'_J}{z'_I} & \text{when} \quad z'_I < 0. \end{cases}$$

(ii) For a change in a_{IJ} within the limits given above, the solution is still optimal, i.e. $\Delta x_j = 0$ $(j = 1, \ldots, n)$.

(iii) For a change in a_{IJ} within the limits indicated above, the value of the criterion function is unaffected, i.e. $\Delta C(\mathbf{x}) = 0$.

Algorithm 6b

Change in a_{IJ} $(J \in \mathbf{B}')$, i.e. x_J is a basic variable in the final tableau.

(i) The minimum possible change in a_{IJ} is given by

$$\underline{\Delta} a_{IJ} = \max \left[\frac{a'_{i0}}{(a'_{iI} a'_{J0} - a'_{JI} a'_{i0})}, \frac{-w'_j}{(z'_I a'_{Jj} + w'_j a'_{JI})}, -\infty \right] \dagger$$

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where the first term is evaluated for all $i \in \mathbf{B}'$ satisfying

$$\frac{a'_{iI}}{a'_{i0}} < \begin{cases} 0 & \text{if } a'_{JI} \geqq 0 \\ \dfrac{a'_{JI}}{a'_{J0}} & \text{if } a'_{JI} \leqq 0 \end{cases}$$

and the second term for all $j \notin \mathbf{B}'$ satisfying

$$z'_I a'_{Jj} < \begin{cases} 0 & \text{if } a'_{JI} \geqq 0, \\ -w'_j a'_{JI} & \text{if } a'_{JI} \leqq 0. \end{cases}$$

The maximum possible change in a_{IJ} is given by

$$\overline{\Delta} a_{IJ} = \min \left[\frac{a'_{i0}}{(a'_{iI} a'_{J0} - a'_{JI} a'_{i0})}, \frac{-w'_j}{(z'_I a'_{Jj} + w'_j a'_{JI})}, \infty \right] \ddagger$$

where the first term is evaluated for all $i \in \mathbf{B}'$ satisfying

$$\frac{a'_{iI}}{a'_{i0}} > \begin{cases} \dfrac{a'_{JI}}{a'_{J0}} & \text{if } a'_{JI} \geqq 0 \\ 0 & \text{if } a'_{JI} \leqq 0 \end{cases}$$

and the second term for all $j \in \mathbf{B}'$ satisfying

$$z'_I a'_{Jj} > \begin{cases} -w_j a'_{JI} & \text{if } a'_{JI} \geqq 0 \\ 0 & \text{if } a'_{JI} \leqq 0. \end{cases}$$

† It can easily be shown that $\underline{\Delta} a_{IJ} > -1/a'_{JI}$ when $a'_{JI} > 0$.

‡ It can readily be verified that $\overline{\Delta} a_{IJ} < -1/a'_{JI}$ when $a'_{JI} < 0$.

(ii) For a change in a_{IJ} within the limits indicated above, the optimal changes necessary in the values of the variables are given by

$$\Delta x_i = \begin{cases} \dfrac{-\Delta a_{IJ}\, a'_{iI}\, a'_{J0}}{1+\Delta a_{IJ}\, a'_{JI}} & (i \in \mathbf{B}') \\ 0 & (i \notin \mathbf{B}'). \end{cases}$$

(iii) The change in the value of the criterion function corresponding to the above changes in the values of the variables is given by

$$\Delta C(\mathbf{x}) = -\frac{\Delta a_{IJ}\, z'_I\, a'_{J0}}{1+\Delta a_{IJ}\, a'_{JI}}.$$

ILLUSTRATION

We will now illustrate the methods of analyses presented in Algorithms 4 through 6 by means of the example problem given earlier with solution tableau given at Table 3. Consider a change in c_5. From Algorithm 4a, a decrease in c_5 will not affect the solution and the solution will change if c_5 increases by more than 50. This implies that the cost of idle time for machine I is not critical at all and it is not worthwhile attempting to find the exact value for this cost. Further, it is profitable to use machine I for an alternative purpose only if the associated profits are greater than \$40.00 per hour. In the case of a change in c_3 (i.e. in the profit per piece of component B manufactured on machine I), from Algorithm 4b the minimum possible value of c_3 without changing the solution is given by

$$8+\max\left[\frac{-4}{4}, \frac{-24}{4}, -\infty\right] = 7.$$

An increase in the value of c_3 will not affect the solution as there is no negative element a'_{3k} $(k \in \mathbf{B}')$. Further, the total profits will increase at the rate of \$720 per unit increase in the value of c_3.

Now consider a change in the time available on machine I, i.e. a change in a_{50}. From Algorithm 5, we have $\Delta C(\mathbf{x}) = 40(\Delta a_{50})$. This means that for machine I it is profitable to have additional hours (say, by paying overtime) at a cost not exceeding \$40.00 per hour. Further, for every unit increase in a_{50}, x_1 has to be increased by two units and x_2 by one unit (with x_3 and x_4 unchanged) if the operation is to be at an optimal level. The minimum and maximum limits where the above figures hold are given by

$$a_{50}+\underline{\Delta} a_{50} = 240+\max\left[\frac{-48}{2}, \frac{-384}{1}, -\infty\right] = 216,$$

$$a_{50}+\overline{\Delta} a_{50} = \infty \text{ as there is no negative element } a'_{j5}\ (j \in \mathbf{B}').$$

We will now consider a change in the production rates. For example, when a_{64} is changed, we find from Algorithm 6a that within the following limits the

values of the variables and of the criterion function are unchanged:

$$a_{64}+\underline{\Delta}a_{64} = 1+\frac{-4}{24} = \frac{5}{6},$$

$$a_{64}+\bar{\Delta}a_{64} = \infty.$$

On the other hand, if the production time per piece of component A on machine II (i.e. a_{63}) is increased, we find from Algorithm 6b that the solution pattern is changed if the production time is more than

$$0{\cdot}25+\frac{4}{(24)(4)+(-4)(4)} = 0{\cdot}25+0{\cdot}05 = 0{\cdot}30.$$

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Corresponding to this increase in a_{63}, from Step (iii) of Algorithm 6b, the change in the value of the criterion function is given by

$$\Delta C(\mathbf{x}) = -\frac{(0{\cdot}05)(24)(720)}{1+(0{\cdot}05)(4)} = -720.$$

Now consider a decrease in the value of a_{63}. Using the equation for $\underline{\Delta}a_{IJ}$, we find that the solution pattern will change if the production time becomes less than 9/40 hours per piece, since

$$a_{63}+\underline{\Delta}a_{63} = 0{\cdot}25+\frac{48}{(-2{\cdot}4)(720)-(4)(48)} = 0{\cdot}25-\frac{1}{40} = \frac{9}{40}.$$

CONCLUSIONS

The algorithms presented in this paper provide convenient methods for obtaining the solution for alternative conditions. This paper can, therefore, be valuable in planning for changing conditions and it will also provide a basis for a more logical decision when alternative proposals are under consideration. Further, when the values of the parameters are not known precisely, the solutions for various combinations of the values of the parameters will facilitate a realistic appraisal of the current operation.

When the constants of the problem are changed, the procedures discussed earlier can result in considerable savings in time and effort in comparison with the alternative of applying the simplex method directly to the changed problem. Further, the algorithms can be programmed on a computer if necessary.

One of the reasons why many of the mathematical techniques have found widespread applications is the ability of the analysts to recognize a possible application. Based on their individual perspective, men from different disciplines using mathematical techniques have brought their own interpretation to a solution. In like manner, application of the algorithms presented in this paper to assess the solution and to use the quantitative information derived from it for taking logical courses of action, will be dependent upon the individual using it.

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Some Industrial Applications of Linear Programming*

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Three examples of the applications of linear programming are given, drawn from the manufacture of plywood, the distribution of grain, and industrial fermentation. In each of these widely different industries there is a problem which can be treated quite precisely by a linear programming model. The quantities evaluated are in the first case, the plywood product mix with the maximum return; in the second case, the routings and transit credits for shipping grain with the minimum freight costs; and in the third case, the production schedule for the fermentation process which yields the minimum labour cost. In most cases, the results obtained could be extended to evaluate other factors, and in all cases the formulation of the problem in linear programming form led to a greater understanding of the industry concerned.

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INTRODUCTION

THE simplex solution of the linear programming problem has been available for over ten years, which is a very long time where operational research is concerned. Following the pioneer work of Dantzig, Fulkerson, Charnes and Cooper, and others, the simplex method has found wide application in the petroleum industry (in which linear programming is commonplace) and the feed industry (manufacturing feed mixes for poultry, hogs, etc.). There are also many other industries in which the applications have been less extensive.

The applications to be discussed here can only be regarded as original in terms of the industry or the peculiarities of the process or price structure handled. They are nevertheless of interest because, in each case, it is a "model" or way of looking at the problem which transforms it into the linear programming form. No new computer programmes were written, nor was any standard programme rewritten (except for the manner of presentation of the output). All the usual teething problems associated with establishing the use of linear programming solutions were encountered and, in the process of overcoming them, more was learned about linear programming itself and the industries concerned. The use of many different computers and programmes in this work led to a great admiration for the ingenuity of mathematicians and programmers.

The industries from which examples will be taken here are:

(1) Plywood manufacturing
(2) Grain distribution
(3) Fermentation.

The plywood example can be regarded as a rather straightforward product mix problem in which joint costs are omitted and the objective is to maximize

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the gross return on available materials subject to plant and market restrictions.

The grain distribution problem would be a straightforward transportation problem except for the long-standing rules and regulations affecting freight rates of U.S. railroads. The initial objective is to meet requirements of customers with shipments via several processing plants at a minimum transportation cost.

The fermentation problem is concerned with a process limited by nature, plant capacity, and labour regulations. The objective is to meet, in each time period, the demands of customers at minimum cost (largely labour). The processes require considerable time, and the limitations of nature are severe. The process takes about 40–50 days, and demands can undergo large variations in that period. Over-aged material must be dumped.

It is interesting that, in all the cases, the linear programming method led, or could lead, to extensions and provide a better insight into the business operations of the industries concerned. This, of course, holds true or should hold true in any operational research study.

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CASE 1. CONCERNING THE PLYWOOD INDUSTRY

Plywood is a sandwich consisting of three or more sheets (or rather surfaces) of veneer held together with glue. Its strength comes from the alternating directions of the wood grains in adjacent sheets. There are more than one hundred plywood products in the standard size (4 ft × 8 ft), differing in thickness, grade of veneer, finish, and type of glue. About 30 products account for 95 per cent of the production of any company. The same product may also be made up in several different ways, known as different lay-ups. Veneers come in 3 or 4 thicknesses, each thickness consisting of 5 to 10 grades. It should be noted that veneer grades and plywood product specifications are quite uniform throughout the industry.

The manufacturer who has his own timber cuts logs from a tract of land and thus obtains a mixture of grades, in fairly stable proportions. The manufacturer who buys veneer does so from a mill which will sell "mill run", that is a mixture of grades, rather than sorted grades. In either case, one cannot attribute a cost to a veneer of a given grade or thickness since it is very much a joint product. Since actual raw material cost is so unsatisfactory, other costs of the production must be considered.

Each product (or lay-up for a product) has well-defined labour and glue costs. An A-grade surface is sanded while a C-grade is not, exterior glue is more expensive than interior glue, pressing costs depend on the product and the type of press used. The supervisory and equipment costs are fixed since the level of activity is relatively independent of the product mix.

To translate the problem into a form to be solved by linear programming, the various restrictions are formulated. For the raw material restrictions, we

can write

$$\sum_i a_{ij} x_i \leqslant b_j. \quad \begin{bmatrix} j = 1, 2, \ldots, M \\ i = 1, 2, \ldots, N \end{bmatrix} \tag{1}$$

where b_j is the fraction of veneer j (specified by grade and thickness) available, x_i the amount of product i produced, and a_{ij} is the amount of material j used per unit of product i. The market restrictions can be specified by equations of the form

$$\alpha_i \sum_{k=1}^{n} x_k \leqslant x_i \leqslant \beta_i \sum_{k=1}^{n} x_k, \tag{2}$$

or

$$\alpha_i' \sum_{k=1}^{n} x_k \leqslant \sum_{i=1}^{k} x_i \leqslant \beta_i' \sum_{k=1}^{n} x_k. \tag{3}$$

Equation (2) states that the proportion of product i to the total output must lie between certain limits. In equation (3) the proportion of a class of products is similarly restricted.

The plant restrictions can be treated in a similar manner. For example, we can write an equation for the restriction on the press as

$$\sum_{\text{interior grades}} t_i x_i \leqslant T \tag{4}$$

where t_i is the press time per unit of product i for, say, a cold press and T is the total available time on the press (set-up times are short).

The objective function which we seek to maximize is

$$Z = \Sigma R_i x_i \tag{5}$$

in units of return per unit of veneer where R_i is the "return" per unit of product i, and is given by

$$R_i = (\text{selling price}) - (\text{trade discounts}) - (\text{product–dependent costs (labour, glue, etc.)}).$$

A fairly large number of calculations has been carried out for three companies, two of which are real, on several different computers. The computation costs for a small (50×100) problem are compared below. A problem of this size arises when, for example, two plants fill a single order book, and the same product made by each of the two plants is considered as two different products.

TABLE 1. COMPARATIVE TIMES AND COSTS FOR COMPUTER SOLUTION

Machine	Time	Cost/hour	Total cost	Remarks
IBM 650	16 hr	\$50	\$800–1,280	No tape
Burroughs 205	2 hr 45 min	\$100–150	\$275–415	Tape
Burroughs 220	5 min	\$300–400	\$30–40	Min. charge is for 1/10 hr
Philco 2000	1 min	\$375	\$37.50	Min. charge is for 1/10 hr
IBM 704	6 min	\$300–500	\$30–50	Min. charge is for 1/10 hr
*Ferranti Mercury	5 min	£75	£7	

* With due thanks to G. W. Sears, Shell International Petroleum Co., Ltd.

It is difficult to compare the computers only on the basis of this time-and-cost basis for such a small problem. Some are more carefully programmed than others and will therefore give greater flexibility or added information (e.g. they will allow partitioning or cost ranging) which may be desirable when larger problems are being handled.

The results obtained are of considerable importance. For a company which is a poor performer, the following comparison can be made:

$Z = \$21{\cdot}16$; existing practice,

$Z = \$22{\cdot}47$; computer solution (no market restrictions).

This difference in return, while a small percentage, is equivalent to more than a doubling of profit; for the plant in question, it amounts to \$150,000 a year. Other studies, including market restrictions, indicate that at least one-third of savings indicated by "ideal" solutions can be achieved in practice.

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The results are significant and become more so when one examines the shadow prices and opportunity costs. In all cases it appears that certain low-grade veneers have larger opportunity costs than A-grade veneer. Also, thin veneers have higher opportunity costs than thicker veneers. This suggests that, contrary to the "good rule" of quality yield, it may be more profitable to downgrade a material than to use it in grade. Certain lay-ups are more profitable than others, and some products have shadow prices (which we might call deferred profits) which are about 50 per cent or more of the return.

In view of these results, the problem can be amended to allow the freedom of downgrading by rewriting equation (1) in the form:

$$\sum_{i=1}^{n} \sum_{j=1}^{p} a_{ij} x_i \leqslant \sum_{j=1}^{p} b_j; \quad (p = 1, 2, \ldots). \tag{6}$$

The results then show that Z can be increased by another \$10,000 per month or more.

There are other by-products of the application of linear programming to the industry. In the long run, the by-products will be of greater value to management than the knowledge of the ideal mix for a given combination of market restrictions, veneer mixes, and selling prices. For example, the shadow prices also indicate factors of value to the sales department. The market restrictions have been set by estimating the company's activity in each geographical area (building codes and local customs define the market areas for specific products, while sales penetration is, of course, also extremely important). One can "rate" salesmen by the "shadow price" of the mix they sell. No one salesman would be expected to sell the perfect mix for the plants, but they should be encouraged to sell a suitable mix for their areas and, if on a commission basis, the commission might well reflect the profit contribution of what they have sold.

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CASE 2. CONCERNING GRAIN DISTRIBUTION

Although this study was carried out for a grain processor, the problem centres around the historical background of railroad rate-setting.

At about the turn of the century, the American railroads instituted a set of "preferred rates" for the processors of agricultural products, with the intention of encouraging such plants to settle along the railroads' rights-of-way. This rate structure was partially encouraged by the government to aid the growth of milling facilities in the Mid-western United States, close to the producers, so that their crops would yield a price comparable to that of Eastern producers.

The rate structure is often referred to as in-process-milling privilege. Subject to limitations, the cost of shipping grain from producer (A) to plant (B) to customer (C) is roughly equal to the cost of a direct shipment from (A) to (C). The limitations are that the privilege does not extend to a backhaul, that the second-stage shipment must go through certain gateways (for a given railroad), and that the second-stage shipment under the privilege must take place within 18 months to 2 years of the original shipment. The privilege is rather general, individual shipments losing their identity; the important quantities are the sum of the shipments from an origin to a plant, and the sum of the shipments from a plant to a customer.

The problem considered was that of a grain processor who has two plants in one region in the United States, receives grain from Western and Mid-western states, and ships to customers between the Pacific Coast and the Mississippi River. Both plants have limited capacity, and all shipments are made in multiples of bulk carloads. There are three classes of finished products, each made from a blend of grains of the same type. (There are 3–6 varieties of a type.) In effect, each customer demands a different product of a class (some only order one class, others 3 classes) which meets his particular specifications.

Let x_{ik} be the amount of product i produced at plant k and a_{ijk} be the fraction of material j in product i produced at plant k, then

$$\sum_{k=1}^{2} \sum_{i=1}^{n} a_{ijk} x_{ik} \leqslant b_j \tag{1}$$

where b_j is the amount of material j available.

$$\sum_{k=1}^{2} x_{ik} = D_i \tag{2}$$

where D_i is the demand for product i (remember a product also indicates a destination). Also

$$\sum_{i} x_{ik} \leqslant E_k \quad \text{(capacity of plant } k\text{)}, \tag{3}$$

$$\sum_{i} Y_{ijk} \leqslant \frac{\beta_k}{\alpha_j} \sum_{i} a_{ijk} x_{ik}, \tag{4}$$

and

$$\sum_{i',j} Y_{ijk} \leqslant \sum_{i'} x_{ik}, \tag{5}$$

where i' is summed over products to the same destination, Y_{ijk} is the transfer credit taken on product-destination i for shipments of raw material j to plant k, β_k is the conversion allowance ratio recognized by the railroad (conversion of grain to product) and α_j is the actual conversion ratio (pounds of product per pound of grain).

The function to be minimized is

$$Z = \sum_{i,k} F_{ik} x_{ik} - \sum_{i,j,k} C_{ijk} Y_{ijk}, \tag{6}$$

where F_{ik} is the freight cost per unit of product via plan k (direct shipping cost from all j's (in product) to customer via the plant) and C_{ijk} is the in-transit allowance (credit) per unit of product i via plant k if taken from origin j.

The problem for two plants and a reasonable number of origins and product destinations results in a linear programming problem of dimensions 76×129 (without slacks or artificial variables). It was run on the IBM 704 (32K core memory) with 6 tapes in about 15 minutes. The routings and transfer credits obtained in the result did not overwhelm management, who found the suggested production and distribution programme perfectly feasible. Among other features of the result were a few products not shipped from the nearest plant (causing some raised eyebrows), and a seemingly extra plant transfer of some imported grain (which does not qualify for in-transit-milling privileges).

A second problem was solved within the 15 minutes of computer time, using a procedure known as partitioning. Since the two plants had just merged, it was felt that production should be split equally between them, and this was the basis of the first solution. For the second solution, both plants were allowed the freedom of running to full capacity. The result showed that one was preferred, at a saving of about \$16,000 or almost \$8 per ton for each ton of grain transferred from Plant (2) to Plant (1). Since the average shipping cost per ton is \$20, the saving is considerable. Further solutions based on the existing solution can be used to determine the value of building additional facilities at the preferred location.

It is hoped to extend this work to include the blending of grains to customer specifications, to include existing inventories of grain and credits, and to evaluate new or altered sources of grain.

A totally unrelated problem will now be considered. Suppose that a manufacturer of, say, a range of related consumer goods (made on the same general equipments) has limited plant capacity. Goods or perhaps subassemblies can be purchased outside, causing a loss in profits, or orders can be refused, also at the expense of profit. The manufacturer can have a number of plants.

Considering the b_j in equation (1) as the capacity limits of the various machines, the x_{ik} as the amount of product i made at plant k and a_{ijk} as the machine hours used for a unit of product i on machine j at plant k, a problem

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of very much the same form is obtained, provided costs and/or profits can be treated as linear quantities. In this case, the Y_{ijk} are the external purchases, more appropriately written as Y_{ix} (or Y_{ixk} if sub-assemblies are concerned). Equations are obtained of the form

$$\sum_k x_{ik} + \sum_x Y_{ix} = D_i.$$

The analogy is not complete, but the treatment of outside purchases within the production matrix and the inevitable limitations on the outside sources, etc. can frequently be treated as a linear programming problem. Further developments in linear programming to include probabilistic demands, and integer solutions, will increase the power and utility of this mathematical model.

It might be re-emphasized here that the models discussed thus far can also be used in some cases to analyse plant location problems for a given distribution, as well as to evaluate equipments and/or raw materials.

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CASE 3. CONCERNING THE PRODUCTION SCHEDULING OF A FERMENTATION PROCESS

Industrial fermentation processes are generally batch operations. The process consists of a number of phases or stages, each of which requires a considerable amount of time. Because of the time factor, the system cannot react quickly to variations in demand, so that the scheduling of production in each stage is of considerable importance.

Consider a fermentation process which has the following stages:

(1) Mixing and cooking	$M, m,$	
(2) Fermentation	$F, f,$	
(3) Purification	$P, p,$	(1)
(4) Blending and packaging	$B, b,$	
(5) Warehousing	$W, w,$	

where m_i (f_i, p_i, etc.) is the quantity that goes into stage m (from the previous stage) in period i; a period may be a week, a day, or even a shift. d_i is the demand (by customers) from the warehouse in period i. The demand d_i is known (but not precisely). Given d_i, a set of values m_i, f_i, p_i, b_i, w_i constitutes a production schedule. The capital letters indicate the total quantity in the stage; i.e. M_j is the total amount of material in the mixing stage in period j.

The basic cost factor in every stage is the labour cost, if only that portion of variable costs which depends on the production rate of a stage is considered. The labour force for each stage is fixed (5 days/week, three 8-hour shifts/day). Extra production requires overtime operations by the crews, which include both

production and "clean-up" workers. Overtime production in a stage will be indicated by a prime (m_i'), and the costs are indicated as follows:

$$\begin{aligned} C_m &= \text{cost/man-hour for straight time labour in stage } m, \\ C_m' &= \text{cost/man-hour for overtime labour in stage } m, \\ C_f &= \text{cost/man-hour for straight time labour in stage } f, \\ &\text{etc.} \end{aligned} \tag{2}$$

A planning horizon is considered to be a large number of periods (i); for a weekly production schedule, the planning horizon is a 3-months' duration, $1 \leqslant i \leqslant N$, where $N = 13$ or 15.

Now consider the process for a daily schedule. The mixing and cooking stage is of fixed duration, t_m periods. Then

$$f_i + f_i' = m_{i-t_m} + m_{i-t_m}'. \tag{3}$$

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Let F_j be the total quantity in fermentation in period j, then

$$F_j = F_0 + \sum_{i=1}^{j} (f_i + f_i' - p_i - p_i'). \tag{4}$$

There is a minimum time required for fermentation and a maximum time allowed by the process. The minimum time is, say, 10 days, and the maximum time is, say, 30 days. Then

$$\sum_{n=0}^{9} (f_{j-n} + f_{j-n}') \leqslant F_j \leqslant \sum_{n=0}^{29} (f_{j-n} + f_{j-n}'). \tag{5}$$

Similarly, for the purification process

$$P_j = P_0 + \sum_{i=1}^{j} (p_i + p_i' - b_i - b_i'). \tag{6}$$

The purification process requires a minimum of 2 days and a maximum of 5 days. Therefore

$$\sum_{n=0}^{1} (p_{i-n} + p_{i-n}') \leqslant P_j \leqslant \sum_{n=0}^{4} (p_{i-n} + p_{i-n}'). \tag{7}$$

Similar relations hold for the warehousing stage. Blending and packaging are carried out on a daily schedule; the total amount passing through the blending stage is restricted only by capacity.

$$\begin{aligned} b_i &\leqslant \text{plant capacity}, \\ b_i' &\leqslant \text{plant capacity}. \end{aligned} \tag{8}$$

Warehousing is limited by the age of the product or warehouse capacity or demand.

$$0 \leqslant W_j = W_0 + \sum_{i=1}^{j} (w_i - d_i) \leqslant \text{warehouse capacity} \tag{9}$$

or

$$0 \leqslant W_j \leqslant \sum_{n=0}^{M} w_{j-n}, \tag{10}$$

where M is the maximum age allowed for storage.

There are restrictions on the available man hours in any stage for both straight time and overtime.

H_{m_i} = max. available hours in mixing in period i, straight time,

H'_{m_i} = max. available hours in mixing in period i, overtime,

H_{f_i} = max. available hours in fermentation in period i, straight time,

etc.

The output expressed in barrels or tons can be converted to man hours by introducing the factors:

μ = number of man hours/unit of production in mixing,

ϕ = number of man hours/unit of production in fermentation,

π = number of man hours/unit of production in purification,

β = number of man hours/unit of production in blending,

χ = number of man hours/unit of production in warehousing.

Then

$$m_i \mu \leqslant H_{m_i}, \quad m'_i \mu \leqslant H'_{m_i}, \tag{11}$$

and mixing cost in period i is given by

$$K_m = H_{m_i} C_{m_i} + H'_{m_i} \mu C_{m_i}. \tag{12}$$

The first term on the right-hand side of equation (12) is fixed for a stable labour force, the second varies linearly with overtime production. Similar relations are obtained for the other stages. The function to minimize is then

$$Z = K_{m_i} + K_{f_i} + K_{p_i} + K_{b_i} + K_{w_i}. \tag{13}$$

This includes the basic labour cost for straight time operations. Since the basic crew is always maintained because of company policy and the need to retain the reservoir of established skills, only the overtime factors need to be included in the cost function.

The problem is now in the linear programming form where the variables are the activity levels in each period, and the restrictions cover the process limitations, the demands to be met, the warehouse and other capacity limitations, and the labour hours available.

SUMMARY

The three problems outlined here are drawn from widely different industries, but in each case there is a peculiarity of the industry which allows the linear

programming form to be a rather precise model for the problem. In many other cases, the peculiarities of the industry would tend to deny the linear form and make this approach impossible.

It is worth noting that, in all three instances, the linear programming study was carried out for companies that were relatively small. The solutions yielded very worthwhile results in real terms, showing that the technique does not necessarily have to be applied to very large-scale operations to be effective.

Experience has shown that merely the process of writing down the pertinent relations in an operation leads to a better understanding of the industry itself as well as of the problem. Further, the process of solution always provides more information than just giving the solution itself. The formulations may therefore change with the passage of time as the operational research worker and the manager learn to understand one another and both learn more about the complex relationships between the variables of the industrial pattern.

The foregoing is not meant to imply that linear programming is the be-all and end-all for the solution of business problems. It is, however, an extremely useful technique, well documented and programmed, with great flexibility, and therefore available for use to high precision when applicable.

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Antithetics: A New Development in Scientific Control*

We have received the following study from a correspondent engaged in management research. It should be read with the slight scepticism that is proper to managing directors when they are offered new aids to the solution of managerial problems.

How is the business executive to cope with his growing army of specialist advisers? Whereas before the war a director consulted his accountant and line managers, he now has to deal in addition with a whole range of experts. Among them are the work study and operational research staff, social scientists and motivational researchers, the ergonomist, the cybernetician and even the colour psychologist. To this bewildering assortment we must now add the antithetician.

Let the managing director of a large industrial company explain; he was, indeed, responsible for the appointment of the first antithetician in this country, and, perhaps, in the world.

"Actually", he told me, "the antithetician began as my PA—terribly bright chap, who always keeps abreast of everything in new ideas, new techniques and new machines. We had a problem about our loading bays at one factory—I was getting really worried about them, for there always seemed to be a long queue of lorries waiting to be loaded. We have three bays altogether, so I said that if we put in another three the queue would only be half as long: that's just common sense, isn't it? Still, common sense is not good enough nowadays, it seems. New capital projects all have to be vetted by the specialists. Of course, everyone passed this one except those work study chaps—or was it that Operational Research fellow? Anyway, someone said that one extra bay would be enough, and that if we put in three the third would be idle most of the time. Obvious nonsense, but they had bits of paper covered with those wretched squiggles they use. It was all beyond me completely, so I had to ask John here if he could prove them wrong."

The first appointment

The antithetician, a slight man with more of the poet than the tycoon in his appearance, took up the story.

"It wasn't too difficult, really. First I saw that they had assumed Poissonian arrivals—an obvious oversimplification. They had also taken a constant loading time, whereas it was obviously a negative exponential distribution. The limits for one of the integrations were rather dubious, and the length of the lorries had been ignored. Oddly enough, they had apparently arrived at

* Reproduced from *The Guardian* of 23 December, 1960, by kind permission of the author and the editor.

the right answer in spite of it all but, as I pointed out to the managing director, one could hardly accept criticism of his project on the basis of such slovenly mathematics."

This, then, was how the company came to appoint the first antithetician, the man whose sole task is to oppose the arguments of any other specialist in his own terms. Like an advocate, he is not concerned with the rights or wrongs of the case, but only with the job of demolishing it. The antithetician has now acquired a department of formidable size, which includes an anthropometrician, an oceanographer, a seismologist and an ecdysiast, each of whom is more than a match for his counterpart in the other service departments. The staff refer jokingly to this group as "The Saboteurs".

One of the attractive features of antithetics is that no problem is too big or too small, too particular or too general, to be outside its scope. Such a startling new advance in management technique is not likely to remain the exclusive property of one company. Although other British firms have shown a regrettable reluctance to adopt antithetics on a large scale (or, indeed, on any scale at all), the idea has spread like a new religion throughout the United States. Most of the larger corporations have set up impressive departments, although across the Atlantic they prefer to be known as Contradiction Counsellors: their recently formed society is already a thousand strong. The new science has progressed far beyond its humble origins, and plans to organize the First International Conference on Antithetics are well advanced: it will probably be held in Paris in the spring.

The president of an important aircraft corporation recently disclosed some of the latest developments in this exciting new field. "We came to realize", he said, "that our Contra Counsellors were in many cases more efficient in their own specialties than the service departments themselves. So now we just change them all over every six months. Ideas are their business, and we tell them to get in there and play those ideas just like a ball game: when you're at the bat, you make runs, but when you go in to pitch, man, you're a Contra!" The British founder of Antithetics is a strong supporter of this system, christened by him "Ambivalent Antithetics". As he says, the House of Commons has used it successfully for hundreds of years, and there seems no reason why it should not eventually be applied to entire boards of directors.

Antithetics is here to stay. The president of the aircraft corporation told me that the ideas rejected by his Contra Counsellors had so far saved $300 millions by eliminating new capital projects. Some confusion was caused when the Contras changed places with the Service Departments for the first time. "Because", observed the great man, "you can't hold those boys. What did they do but prove straight away that those $300 millions hadn't been saved at all. We soon worked that one out—just a short course in Positive Thinking down in the cellars—and they're shaping up real well. Why, they've just reinvented the biplane."

Letter to the Editor

WHEN operational research workers meet, there is general agreement to condemn the literature; the papers published do not communicate that part of an operational research job that the operational research worker finds satisfying, but consist of exercises in applied mathematics. I want to add to the plethora of excuses for this failure what I believe to be a reason rooted in the nature of operational research activity.

Consider the archetypal operational research job. The operational research worker is introduced to an operational situation in which the decision-maker is finding difficulty in making decisions and in which the ends existing in the situation are uncertainly apprehended, and rarely explicitly stated in a complete form. The first part of the operational research job consists of understanding the true nature and purposes of the operations considered, from experiment and records, by the process of articulation in the operational research worker's mind of a valid model in which the objectives of the operations can be adequately stated. He is now in a position to advise the decision-maker by telling him of the true ends and will usually be able to give a method of achieving a possible, a near optimal, or an optimal decision, depending sometimes on simple and sometimes on highly sophisticated mathematical, logical or analogical techniques.

Now if the time comes to communicate the operational research study to those unfamiliar with the original situation, it is straightforward to write up the second part of the job, the mathematical justification of the methods of decision-making advised, but the value of the work to the operational research worker lies in the first part and there is a logical barrier to communicating this.

For what must be communicated is the operational situation, as distinct from the model that is the operational research worker's view of the operational situation. But the language the operational research worker has developed to describe the operational situation is precisely the model of the situation that we must distinguish from the situation itself. The impasse is total. This part of the operational research job is not communicated and another article is printed for operational research workers to lament.

Relief from this vicious circle is possible only to those who have been working on public service problems where the operational situations may already have been felt by the reader as participant. On the one hand there are studies of traffic congestion, on the other of the optimal pursuit of loans.

Where are the other avenues of escape?

J. LASKI

Cybor House,
1 Tapton House Road,
Sheffield, 10.

Book Reviews

The Statistical Basis of Quality Control Charts—a manual for business and factory managers.

S. K. EKAMBARAM.

Asia Publishing House, London. 16*s.* 6*d.*

In 18 short chapters comprising 93 pages the author gives an introduction to statistical quality control, written, in his own words, "for the busy executive in the higher levels of management". After a brief philosophical dissertation on the role of statistical quality control in modern industry the properties of the normal distribution are fairly fully discussed and the construction and use of $\bar{X}$, R charts explained. The Poisson distribution is then described and the two kinds of chart used for control with qualitative data discussed. The main points of each kind of control procedure are illustrated with actual examples taken from Indian industry. There is a final very brief chapter on acceptance sampling and the operating characteristic curve.

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The excuse for reviewing this book in an operational research journal must be, I suppose, that many operational research groups are still intimately connected with industrial statistics and are often called upon to advise on quality control schemes. In addition, of course, the proposed solutions to operational research problems must themselves sometimes be subjected to a statistical control.

Three questions can therefore be asked of the book: Does it fulfil the author's intention as quoted above? Does it aid the operational research worker in his occasional role of industrial statistician? And thirdly, does it contribute anything to the problem of control in industry in the wider sense? The answer to the first question is, No; to the second, Maybe; and to the third, No. The busy executive, for whom the author ostensibly writes, would, I feel sure, find the introduction conceptually vague and somewhat patronizing. He would also find that although the chapters follow a logical sequence, they are not very well linked together. For example, the chapter on frequency distributions follows a definitive chapter in which a process is described as being in control when all its products belong to the same population in some defined characteristic. The definitive chapter is written purely in conceptual terms and the frequency distribution chapter deals immediately with number. A mental adjustment is therefore required by the reader to link the two together.

Many terms are used without definition at all or without adequate definition to someone unfamiliar with statistical parlance. Examples are: efficiency, inter-quartile range and standard deviation. The early mention of infinitesimal calculus and the amount of mathematics used to describe properties of the

normal and Poisson distributions which are not used in quality control work are not likely to enamour the book to the busy manager. In short, the book is too mathematical and insufficiently discursive in exposition for the author's prime purpose to have been achieved.

Is the book of much value to the sophisticated operational research man, then? As a fairly quick refresher of some of the main statistical points of quality control it has its uses, although again one has the impression that the author has tended to put down what he knows rather than what the reader needs to learn. There is a useful section in the chapters on the Poisson distribution in which a clear distinction is made between qualitative information as a measure of defectives per batch as in, say, the manufacture of induction coils and where the measure is defectives per unit as in, say, the number of flaws on a photographic film. This distinction is not always made as clearly in quality control literature.

The author has been a little misleading, however, in assuming that, with qualitative measurements, all process parameters have normal distributions. This carries the implication that normality is necessary for statistical quality control work. He makes no mention of the important property that, even if the basic distribution is not normal, the average of samples of, say, five may be so nearly normally distributed as makes no difference. The control limits used in the $\bar{X}$, R charts are only the 3σ limits. No use is made of the inner 2σ warning limits. The $\bar{X}$, R charts are relied on exclusively for process information. No mention is made of the advantage to be gained by occasional plotting of the frequency distribution of the whole process.

The book thus only conveys a limited experience to someone with the necessary mathematics but no previous knowledge of statistical quality control work.

On the wide question of industrial control the author says nothing.

There are some misprints in the book, the grammar is not impeccable and the symbol for infinity is not the usual one. These are minor irritants in a disappointing book.

W. E. DUCKWORTH

Introduction to Linear Programming.

W. W. GARVIN.

McGraw-Hill Publishing Co. Ltd., 1960. xix+281 *pp.* 68*s.*

The book by Charnes, Cooper and Henderson of the same title (apart from an indefinite article) appeared in 1953 and was the first of a series of volumes which deal with this subject, but take account of more and more topics which have joined it. The latest book is fairly up-to-date (1960), and the title merely means that you cannot find everything in it.

The book does, in fact, not deal with theoretical aspects more than is necessary for the practical man to understand what he is doing. Computational aspects are stressed, but apparently for the benefit of the worker who does his sums on paper. Good flow diagrams are given and will be useful to those who wish to write computer programmes. The exercises at the ends of the chapters are welcome, but more of them (with solutions) would have been better still.

On the whole the contents are those that you would expect in a book on this subject. The three Parts are entitled: The General Linear-Programming Problem; The Transportation Problem; Special Methods. Topics not found to the same extent in other books are a discussion of the dependence of the solution on the technological coefficients, the generalized transportation problem

$$\sum_j x_{ij} = a_i, \quad \sum_i e_{ij} x_{ij} = b_j, \quad \text{minimize } \sum_i \sum_j c_{ij} x_{ij},$$

then upper bounds to the variables, and statistical linear programming. Kuhn's Hungarian Method is not mentioned. The Ford–Fulkerson Method is mentioned, but not described.

The explanations are lucid and the text is most readable. The author says that the book has been limited in size to make it suitable as a text-book for a one-semester course and it serves this purpose well.

S. VAJDA

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Economics and Operations Research: A Symposium, followed by "An Economist Looks at Air Force Logistics", S. ENKE.

Review of Economics and Statistics, 1958, **40**, *pp.* 195–239.

This extremely interesting collection of papers is devoted to the relation between economics and operational research. The general picture the authors give is that, in general, operational research problems are economic problems. In the past, economists have developed considerable insight into criteria of optimality, but they have shown too little awareness of the complexity of real life, and their tools of analysis and measurement have remained primitive. Operational research workers, mostly mathematicians or natural scientists by training, have been far readier to take complexity into account, and to develop powerful tools of analysis, but they have at times shown themselves regrettably naïve about choice of criteria. Thus there appears to be scope for mutual help and education.

(1) W. W. Cooper (Operations Research and Economics) treats operations research as "management engineering". He points out the immense power of the approach of the natural scientist to real-life problems, but indicates the need for the insights of the social sciences (including economics) in the study

of organizations. He discusses the way in which some training in these might be given.

(2) C. Hitch (Economics and Military Operations Research) points out the value operational research has had in exhibiting the wide range of alternatives open to the policy-maker, and in devising tools to deal with such ranges—though he feels that the economists' concept of substitutability should probably extend the range even more widely. He attacks the naïve choice of criteria of optimality in some published work (specifically a case quoted from Morse and Kimball) and points out the desirability in many cases of a difference (profit or loss) as against a ratio criterion. He points out the need for a joint effort by operational research workers and economists to build a welfare economics for a system without markets.

(3) W. J. Baumol describes two cases where operational research can yield valuable information and analytical tools for the economist.

The arguments are further developed by T. C. Schelling, S. Valavanis and D. Ellsberg.

Enke's paper gives a detailed description of the supply system of the U.S. Air Force and the economic (or operational research) problems it throws up.

T. E. EASTERFIELD

Introduction to Congestion Theory in Telephone Systems.

R. SYSKI.

Published for Automatic Telephone and Electric Co. Ltd., by Oliver and Boyd, Edinburgh, 1960. xvi+742 *pp.* 105*s.*

Apart from an introductory chapter, and a chapter on telephone systems (which is not by the author of the rest of the book), this is a book for mathematicians. Telephone engineers, who are the only potential readers named, must be exceptional if they can read it without a great deal of hard work. Although Chapter 3 is on Probability Theory, a prior knowledge of this subject is really necessary. The structure of the book, naturally, is on the basis of telephone systems, but the 100 pages of Chapter 6, on Waiting Systems, deal with queueing problems as normally understood in operational research. Chapter 9, on Special Systems, deals with unusual queue disciplines and a miscellany of special problems and approaches. The remainder of the book is on the congestion theory of other telephone systems and is less likely to be of direct value outside the telephone field.

The book is thorough in the sense that it gives most of the possible approaches to congestion problems and covers the major contributions to the subject. The heaviest mathematics, such as those of Pollaczek's work, are not included. It is not very well written, but this becomes less noticeable as the ratio of

mathematics to words increases. One feature of the writing is the extensive use of abbreviations—"The most striking property of the n.e.d. concerns the pr. of the prolongation of the time interval t".

To sum up, much of this book is particular to telephone systems. On the other hand, there is so much in it that the proportion which is of more general interest is still a fairly substantial amount. The mathematician wishing to delve reasonably deeply into queueing theory could find it a useful starting-point because of its coverage of most of the notable published work in the field.

R. A. ACTON

Management Dictionary: English—German.

W. SOMMER and H.-M. SCHÖNFELD.

W. de Gruyter & Co., Berlin, 1960. 176 *pp.*

This book, attractively presented in a pocket-size, with covers of a plastics material, is not what it says it is. It is not a dictionary but a vocabulary or glossary of equivalent terms in the two languages specified. Even the English is rather American-English, and there are many terms given from American practice which have their direct counterparts in other English-speaking countries but which are omitted. This may befit the Introduction by the Professor of Accountancy at the University of Illinois, but it certainly cannot fully reflect ". . . the situation in the English-speaking countries . . .".

There are many words of quite common usage which are included in this glossary which purports to be of management terms. Indeed, some of the entries are such as to put one on inquiry as to the concept of management in the minds of the compilers. The inclusion of such words as:—

innocence, acid test, halo effect, homework and perk-up

are a few of the many that could be cited. There are other pieces of evidence that cause some concern over the value of this glossary. Is there really a "gray" market? Something should not be "inofficial", and an electric current is alternating rather than "alternate". Goods are often delivered free on quay rather than on a "quai", and the manager who tries to express the quite ordinary legal phrase of "Last Will and Testament" will find some repetition in the German equivalent. In the field with which this reviewer has some special experience it is clear that the compilers of the glossary under review have paid but scant attention to existing glossaries of equivalent terms or are unaware of their existence and the problems involved.

While it is important to promote the interchange of ideas between the major language-groups of the world, the task of compiling the appropriate glossaries of terms is by no means easy. Indeed, it is probably one which needs to be

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carried out under the auspices of the relevant international body in order to avoid the difficulties and pitfalls which must beset the private investigator.

WM. R. BUCKLAND

Les Choix Economiques—décisions séquentielles et simulation.

P. ROSENSTIEHL and A. GHOUILA-HOURI.

Dunod, Paris, 1960. 355 *pp.*

The intended scope of this collection of papers is both comprehensive and ambitious. The volume as a whole is intended for those who are concerned with the taking of economic decisions which should be both spatially and temporally coherent under complex conditions of uncertainty.

After a discussion of generalities given in French and in English, the two principal authors give a useful introduction to sequential techniques and to simulation methods. There are three chapters in French discussing firstly dynamic programming and then the nature and simulation of stochastic processes with an indication of the contribution that machines can make in such studies.

In the manner of recent conferences this part of the book is followed by a series of applications given in English or in French by a variety of authors. The quality of these contributions is uneven. There is a tendency towards preoccupation with the details of the subject of the application rather than with detailed illustration of the methods described in the first few chapters. This is perhaps a pity in a book of this kind—the scope of the methods is made clear but a somewhat disjointed impression is created by the treatment of the various subjects which include air traffic control, logistics policy, the control of river waters, war games and business games.

English readers will at times find the language of the translation of the first chapter a little curious. For the rest of the book they will have to work for half the time in French. There are good bibliographies but only a rudimentary index.

D. R. KAYE

An Introduction to Infinitely Many Variates.

ENDERS A. ROBINSON. (Edited by M. G. KENDALL.)

Griffin's Statistical Monographs and Courses.

The subject of statistical analysis of time series is one which the operational researcher cannot avoid these days and in this field the methods of spectral analysis play a part which is strictly analogous to the part played by the histogram in the statistical analysis of independent observations. The parallel between the histogram and the spectrum goes very deep—the problem of "smoothing"

the spectral density function is of essentially the same character as the problem of "smoothing" the histogram. We solve the latter, usually, by a suitable choice of grouping interval, making it not so wide as to obliterate all the ups and downs in the distribution, nor so narrow as to be too much affected by merely random fluctuations. Exactly similar considerations apply to the spectrum, only we have to operate at a slightly higher level of sophistication in which some of the more refined aspects of the smoothing process are taken care of. If the histogram had not been in common use in elementary statistics for many years, and were to be invented to-day as a new tool, one could imagine that for a long time the essentially simple subject matter involved in discussing it would be overlaid by enormous masses of mathematical apparatus, and we would have to wait a long time for the theory to stand out in its essential simplicity. The theory of spectral analysis has in effect suffered this fate. It is only gradually emerging from a morass of irrelevant higher mathematical detail.

Dr. Robinson's book represents a stage in the emergence of spectral theory from the matrix of higher functional analysis in which it has been so far embedded. The list of topics covered includes a general limit theorem, Hilbert space, linear operators, spectral representation and stationary processes. The more fundamental subjects are dealt with by way of very terse accounts of the theory, supplemented by exercises such as the exercises 5.4: "Write a paper on the connection between linear operators and infinite matrices in Hilbert space", or 5.5: "Develop the properties of the stochastic integral. Find applications in the theory of stochastic processes, for example in the theory of Brownian motion".

As a handbook giving a rapid conspectus of the abstract theories involved, Dr. Robinson's work is useful, but it is to be doubted whether it will serve the purposes of those who wish readily to understand the new tools, for whom a definitive and simple account remains to be written.

G. A. BARNARD

Control Systems Engineering.

WILLIAM W. SEIFORT and CARL W. STEEG (Editors).

McGraw-Hill, New York. 964 *pp.* £5 16*s.* 6*d.*

The justification for the title is not clear since this book consists entirely of some mathematical techniques useful in the design or analysis of large servo-mechanical systems. Examples of their use in the real world of engineering were excluded on the grounds that the size of this book required limiting. This is unfortunate since many text books on techniques have been written, but few indeed on their utility.

The first half of the book is a brisk canter through the basic mathematical theory, starting with Ordinary Differential Equations, through Numerical

Analysis and limping home on Engineering (*sic*) Applications of Statistical Theory. (Why must it always be assumed that the purchaser of such books is either mathematically ignorant, or too indolent to refer to a mathematics text book?) This section, however, is clearly written and forms a useful, albeit lengthy, aide-memoire.

The second half covers the use of these techniques in the analysis of typical, idealized, servomechanical systems. This is yet another manual dealing principally with linear, or near linear systems. This includes an account of Sampled-data Systems which, with the advent of process control computers, is an expanding field. For a not very discernible reason, the editors have included a chapter on the Elements of Game Theory which makes the omission of articles on Self-adaptive Systems rather surprising. This field is surely the one on which servo-control has the brightest future and a new area of applications.

This is certainly not the book from which the operational research worker can ascertain what are the current developments or what techniques are practicable in the servomechanical-control field, over-served as he is with manuals of mathematics.

P. J. O'DONNELL

Analogue Computation.

STANLEY FIFER.

McGraw-Hill, New York, 1961. xxi + 1,331 *pp.* (4 *vol.*). £15 6*s.* 6*d.*

Solution of physical problems by means of computers is commonplace but as readers of this quarterly may have used the digital computer more than the analogue, this review provides an opportunity to survey the potentialities of the electronic analogue computer. With any computer, the first step is to formulate a mathematical description of the problem; then, with an analogue computer, a physical system obeying the same equations is built up and the solution is obtained by making physical measurements on it. Engineers like this method which is noted for the physical insight into the problem which it can give—it is very similar to working with a model.

Fifer is associated with the manufacture of electronic analogue computers and is obviously experienced in their use. His four volumes are well written and describe the computer, the formulation of mathematical descriptions, the methods of setting them up on the analogue, taking measurements and checking. A large range of applications is described but the book is not specifically intended for operational research users.

In the computer, numbers are represented by the magnitudes of voltages, resistances, capacitances and time-intervals. (Time is the independent variable when integrating.) After the introduction, he describes the chopper-stabilized

D–C amplifier (used with resistors and capacitors for addition and integration) and linear potentiometers (for multiplication by constants). He then shows how to solve linear differential equations with constant coefficients. Then follow chapters on multipliers (the important Hall-effect multiplier is omitted), on resolvers (for trigonometry) and on diodes and relays (for setting limits on variables, for simulating "backlash" and for detecting maxima and minima). That actually ends the first volume, but the division into four equal volumes has no logical counterpart in the text.

Curve-plotting devices and digital voltmeters are recommended for extracting information from the computer: the former give an overall picture and the latter waste none of the accuracy attainable in the more expensive computers. Later on, when dealing with the faster but less accurate "repetitive" computers he omits to mention the 3-dimensional displays used by Mackay to present large amounts of information in a form that the eye can readily use.

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Function generation is described at length so that he can deal with differential equations which are non-linear or have non-constant coefficients. Examples include the Bessel, Legendre and van der Pol equations and the author reveals an interest in missile and target kinematics in other examples. There is a chapter on transfer functions and one on checking and error analysis.

A large part of Vol. 3 deals with other analogue devices: the mechanical differential analyser, the network analyser, finite-difference network analogues of partial differential equations, the electrolytic tank. Brief notes on such devices as mechanical harmonic analysers will also be found. These devices are described to enable comparisons to be made with the electronic analogue computer applied to the same problems and to reach, by implication, the conclusion that the latter can tackle an enormous range of problems if a large installation is available. The mathematical-physical principles of these other analogue devices are clearly presented but practical details are better sought elsewhere. The treatment of analogue devices using transformers (largely a British development) is incomplete.

The applications of the electronic computer include: ordinary differential equations (already mentioned), linear or non-linear partial differential equations (using finite-difference approximations for all but one of the independent variables and time for the other), solution of simultaneous, linear algebraic equations (at least n amplifiers and n^2 potentiometers are needed for n equations; to avoid computer instability, more may be needed), solution of polynomial equations (including steepest-descent methods), latent roots of matrices, Fourier analysis. Linear programming is not mentioned but as practical problems usually involve a large number of variables, the digital computer is more suitable; the analogue computer may be useful for preliminary studies if any non-linearities are introduced into the problem.

Volume 4 is mainly concerned with more technical applications: noise (theory, correlation computers, weighting functions, noise generators), aerodynamics

(flight dynamics, flight simulators and trainers, flutter), production of time-delayed signals (important in chemical plant simulation), simulation of sampled-data systems, applications to economics and automation. The author's special interests result in this last topic having only 13 pages whereas aerodynamics has 136.

The book is recommended to actual and potential users of electronic analogue computers (if they can afford it) and to managerial staff wondering whether to buy one. The mathematics is particularly clear. The only serious error noticed is in Fig. 2.16 (d) where the amplifier input is earthed. Many exercises are given and there is a large number of references ranging from Lord Kelvin's papers on the principles of a differential analyser to an article entitled "Will the Logger-Computer Optimize Refining?"

J. R. BARKER

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Kybernetik: A Journal dealing with the Transmission and Processing of Information as well as with Control Processes in both Organisms and Automata.

Edited by H. B. BARLOW, Cambridge, England; M. HALLE, Cambridge, Mass.; B. HASSENSTEIN, Freiburg i. Br.; W. D. KEIDEL, Erlangen; I. KOHLER, Innsbruck; K. KÜPFMÜLLER, Darmstadt; H. MITTELSTAEDT, Seewiesen, Obb; W. REICHARDT, Tübingen; W. A. ROSENBLITH, Cambridge, Mass.; J. F. SCHOUTEN, Eindhoven; M. SCHUTZENBERGER, Poitiers; K. STEINBUCH, Karlsruhe; N. WIENER, Cambridge, Mass.

Springer-Verlag, Berlin. Vol. I, No. 1. *DM.* 12,80.

The first issue of this new journal, in 56 pages, of which 9 are in English and the rest in German, has appeared. The proposed method of publication seems to be an attempt to discover cybernetically how often the journal ought to appear. An issue will be printed as soon as possible after receipt of manuscript material; each issue may have a different size and price, and only a maximum subscription price for the year (about £7 10*s.* 0*d.*) is quoted.

Of the thirteen so-called editors, Küpfmüller, a professor in the Technical University of Darmstadt who calls his research effort an "Institute for General Communications Technics", and Reichardt, of the Reichardt Division of the Max-Planck-Institute for Biology in Tübingen, are singled out as recipients of submitted manuscripts.

Not all the articles are provided with summaries. This is a most regrettable omission.

Since it has been alleged that there is a special connection between operational research and cybernetics, the question "is this journal of interest to people in operational research?" should not be answered hastily. The contents of the journal form a fair random sample of cybernetic activities, and the journal may therefore be used as a test of the hypothesis that there is a special connection.

The following is a short indication of the contents.

The leading article by Küpfmüller and F. Jenik is a brief account of experiments carried out in the Institute in Darmstadt with "a kind of analogue computer" using a pulse technique with variable frequency and amplitude to represent a system of nerve cells, each cell having multiple activating and inhibiting inputs, thresholds, provision for holding and summing inputs over time, and for a recovery period after firing. Relationships between the output frequency and various combinations of input amplitudes and frequencies are reported. F. Wenzel, of the same Institute, reports measurements on recognition times for reading music and letters.

K. Steinbuch, another professor (in Karlsruhe) who has chosen to call his research team a Communications Institute, gives a general description of "The Learning Matrix" on which his people are working. He hopes that it will be applicable in automatic sign recognition and information retrieval. The Matrix forms "conditioned reflexes" by associating signals from an "eye" (six photo electric cells each seeing a field which is either black or white) with "meanings" (positions of switches).

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It is claimed that the Learning Matrix provides a model in terms of which it is possible satisfactorily to distinguish the concepts "signal" and "information", this being impossible in information theory as at present known, which is sterile for want of a semantic element.

Reichardt has a longer article on growth—reactions to light in phycomyces. D. Trincker, J. Sieber and J. Bartual, all of Erlangen, describe the recording of eye movements stimulated by Pendulum movements of the subject, or of the visual field, or by electrical currents across the head. The results are said to support the hypothesis that eye movements are programmed from a common source which is activated by information input in diverse afferent ways.

G. Vossius (Frankfurt-am-Main) reports some purely theoretical work on the control circuit governing muscular movement, D. H. Casson of Johns Hopkins writes on letter constraints within words in printed English, and there is a note describing a simple electrical demonstration model for Pavlov's dog by I. Kohler (Innsbruck).

Readers engaged in operational research can now form a provisional impression for themselves as to whether cybernetics as represented in this journal concerns them in a special way. A few observations may be offered. First, it is obvious that a great deal of detailed work must go on, for instance in the biochemistry of the information systems inside living organisms, which an operational research worker can hardly be expected to keep up with step by step. Second, if a new model is developed such as is promised in Steinbuch's semantic information theory, this would be important for an operational research worker. Third, information and control are commodities with which operational

research is seriously but by no means exhaustively concerned. So far, there is nothing to distinguish the operational research man's obligation to be in touch with cybernetics from his need to be in touch with many other sciences which provide him with models and have a content which overlaps with the content of his work. This issue of *Kybernetik* with its emphasis on physiology and computer-like technologies cannot claim a special relevance to operational research.

P. G. LUCAS

Computer Abstracts.

Technical Information Co. Ltd., London. Annual subscription £25 4*s*. 0*d*.

The aim of this journal is to provide a comprehensive and well-referenced record of abstracts from literature published throughout the world on computers and the techniques of their use.

The vast amount published on these subjects has necessitated a compromise in the length of these abstracts between an extract too short to be of any real value, and a length which would make each volume too large and destroy its usefulness as a quick guide. This compromise has been happily effected and the journal reviewed achieves its aim.

The contents are split into two main sections. One deals with the theoretical and technological aspects of computers and is subdivided under various headings, e.g. Design, Mathematics, Digital Computers, etc. The second section is concerned with Applications, subdivided into various professional, industrial and academic sections.

Each issue also contains alphabetical name and patent indexes, cross referenced to the relevant abstract, plus an insert Computer News. This gives information on exhibitions and conferences and news from computer users and producers.

In addition, the publishers offer a document procurement service for all subscribers to the journal.

The printing is very clear and each page is printed on one side only to facilitate the filing of abstracts.

The subscription rate seems high, and is no doubt caused by the amount of work in research and translation involved in the regular scanning of over 600 scientific and business publications, 8500 patents from Britain, Germany, Russia and the U.S.A., and the systematic classification of over 500 references a year.

W. A. ALLISON

Self-Organizing Systems.

MARSHALL YOVITS and SCOTT CAMERON, *Editors.*

Pergamon Press, London, 1960. viii + 322 *pp*. 60*s*.

A book bearing this title ought to contain genuine cybernetics, a discussion of the formal principles which make real systems viable and survival-worthy.

Expectations of this kind are often frustrated; the sanguine reader encounters instead a mass of computer technology leading to a model of an artefact of a real system. It is therefore especially pleasurable to declare that this book *does* contain genuine cybernetics.

These are the proceedings of an interdisciplinary conference sponsored by the American Office of Naval Research and the Armour Research Foundation, and held in Chicago in May, 1959. Fourteen papers are printed (with associated discussion) and arranged, somewhat arbitrarily, in four groups. They comprise a most interesting and valuable book.

Dr. Weyl opened the conference with the sally: "Good morning, self-organizing systems". Had he then sat down, he would at least have provided a useful text. For the whole point about complex systems that exhibit a high degree of self-organization is that there are such things in nature. If we want to know about this vital phenomenon we have only to look around ourselves and into ourselves. Do we want to know? Considering the degree of disorganization in the artificial systems we have constructed (industrially, economically, socially), we do. Considering the theory of control we have so far mastered, we do. For it is given in the results of such workers as Shannon and Ashby that there is a necessary connection between the amount of complexity in the system and in its controller; these amounts must match if the system is to undertake advanced behaviour (rather than to die, for example). The amount of external apparatus that must be clapped on to a really complex system, a company perhaps, in order to control it from outside is unthinkably vast. The only sensible thing to do is what nature does: cause the system to control itself; make it absorb the variety generated in one mode of its behaviour with that generated in another; make it *self*-organizing. This is indeed how many artificial systems are controlled. The question is whether scientific research can improve the insights on the basis of which such management is undertaken.

The first point to note is that a self-organizing system must be richly endowed with input: a necessary condition for the matching of environmental with controlling variety is adequate channel capacity to link the system to the world. Thus it is proper for the first group of four papers to deal with *perception.* This requires the self-organization of incoming sensations into patterns. How such neurophysiological procedures may be studied formally is discussed by Farley. Estes develops statistical models for the recall and recognition of these patterns, and Rosenblatt considers the mechanism for their generalization over transformation groups. A particularly thought-provoking paper by von Foerster treats the interaction of system and environment through the measure of entropy, using a thermodynamic model.

By now the conference is moving naturally into a second phase, dealing with the effects of environmental feedback. von Foerster had illustrated his paper with a trick whereby magnetized cubes organized themselves after an environmental shaking. What happens *in vitro*? Auerbach, an embryologist, explains

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how embryonic cells organize and reorganize in these circumstances, and describes some fascinating experiments. Goldman (electrical engineering) attempts the analysis of the body's homeostat for controlling glucose, and Bishop continues by assessing the role of environmental feedback in brain function itself. The discussion contains an interesting account of the results of sensory deprivation in man, which underlines the importance of perception and interaction to a viable system.

The third section turns to a particular problem of self-organizing systems: that of learning. Newall, Shaw and Simon contribute a lengthy account of their GPS (general problem solver) which is a heuristic computer programme inherently capable of self-improvement, while Milner returns to the natural archetype of learning—the neural system. Campbell urges the importance of random mutation to any self-organizing system from the viewpoint of a psychologist.

In the final section come four papers on the structure of these systems. Pask approaches the description of networks from the standpoint of the natural historian. McCulloch explains his redundant neural nets as offering the ultimate guarantee of reliability in systems made of unreliable components, which is a very important concept indeed. Turning from the probabilistic behaviour by which the other symposiasts have characterized viable systems, Burks discusses the structure of deterministic automata. Uttley concludes the conference with a banquet address on the mechanization of thought processes.

This brief guide may leave unmoved those operational research men who are happy with the notion that they can systematically create an adequate description of a complex system and its organizing mechanisms by inventing rows of equations. But they ought to read this book, bearing in mind that in a natural system such as the brain there are 10^{10} neurons interacting—each of which (some claim) needs an eighth-order non-linear differential equation to describe its behaviour. Industries are, perhaps, no less complicated than brains, and it may be that other lines of approach are needed. Those already satisfied on this point will not wish to miss a most important book which indicates many such possibilities.

STAFFORD BEER

Abstracts

Simulation of a Full-scale Multi-stage Batchwise Chemical Plant.
P. V. YOULE.
The Computer Journal, **3**, 3, pp. 150–157 (October, 1960).
A simulation programme written in Mercury Autocode has been used to study a typical, hypothetical chemical plant consisting of seventeen vessels in four stages, engaged on batch production. The results indicate ways of raising efficiency and output.

R. H. M.

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A problem of Delayed Service—I and II.
A. BEN-ISRAEL and P. NAOR.
Journal of the Royal Statistical Society, Series B, **22**, 2, pp. 245–276 (1960).
A number of machines are each liable to break down, and a repairman tends to them in a prescribed order, some of the repairman's time being taken up with ancillary duties, such as walking and inspection. A variety of models are discussed, making varying assumptions about the distribution of breakdowns, and the attendance time distribution of the repairman. Paper I considers only successional or simultaneous attendance to machines. In paper II, the results are extended to a mixed assumption, that the repairman tends to a group of machines simultaneously, the group being dealt with in succession.

Two basic assumptions made are that attendance time of a machine by a repairman is independent of whether the machine is working or idle, and that breakdowns possess Poissonian character (this is defined in two ways). Expected values and variances are derived for quantities such as the number of idle machines, and the waiting time of an idle machine.

G. B. W.

On the transient behaviour of a Simple Queue.
P. D. FINCH.
Journal of the Royal Statistical Society, Series B, **22**, 2, pp. 277–284 (1960).
We consider a single server queue with Poisson arrivals and general distribution of service time and seek to determine the probability distribution of queue size after the nth departure from the system. A generating function is obtained which solves this problem formally. When the service time distribution is negative exponential an explicit solution is obtained.

G. B. W.

Queuing at a Single Serving Point with Group Arrival.
B. W. CONNOLLY.
Journal of the Royal Statistical Society, Series B, **22**, 2, pp. 285–298 (1960).
A simple single server queueing system is considered, with negative exponentially distributed service time, and general-independent type input in which each arrival consists of a fixed number of individuals. "Steady state" formulae are derived relating to the queue size, the busy periods, etc.

G. B. W.

The Logistic Process: Tables of the Stochastic Epidemic Curve and Applications.
E. MANSFIELD and C. HENSLEY.
Journal of the Royal Statistical Society, Series B, **22**, 2, pp. 332–337 (1960).
The logistic process is a simple stochastic process which provides a model for such phenomena as the spread of an epidemic in a closed group of n persons. Interest centres in how the distribution of the number of "infected persons" varies with "time". Tables are provided from which the rate of change of the mean number of infected persons can be read for a series of values of "time", for $n=5(1)40$: by numerical integration, the mean or other moments of the distribution can be obtained.

G. B. W.

Approximate Solutions of Green's Type for Univariate Stochastic Processes.
H. E. DANIELS.
Journal of the Royal Statistical Society, Series B, **22**, 2, pp. 376–401 (1960).
A study is made of approximations of Green's type (often called W.K.B. approximations) to the solution of equations governing univariate stochastic processes. After considering some specific processes, the general homogeneous process in continuous time is discussed. A numerical comparison is made between the approximate and exact probabilities for the multiplicative birth and death process. Diffusion processes are briefly examined. The method is applied to the equations both for probabilities and for the moment general function, and the duality between the approximate solutions is indicated. Finally, approximate formulae for the cumulants are obtained and the limitations of the method are pointed out.

G. B. W.

Linear Theory.
J. R. HICKS.
Economic Journal, **lxx**, (280), 671–709 (1960).
Examples are given of studies undertaken by the Operational Research Unit of the British Transport Commission. They include an early study of the design of

passenger stations and investigations into various aspects of freight services. Reference is made to the development of systems for recording wagon movements and to the use of linear programming and simulation techniques.

P. W.

Operational Research in British Transport.

M. G. BENNETT.

British Transport Review, **vi**, (1), 3–15 (1960).

This is an expository paper, written for economists. It covers the following topics: linear programming, activity-analysis, input–output analysis and the theory of games. It includes non-mathematical accounts of the diet problem and the transportation problem and a discussion of their relevance to economic theory. There is a brief discussion of the scope for using game theory in economics.

D. J. B.

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News and Notes

OPERATIONAL RESEARCH SOCIETY
ANNUAL REPORT FOR THE YEAR ENDED 31 JULY, 1960

THE Society's winter meetings were held in the lecture theatre of the Royal Society of Arts, with the exception of the 8 March, 1960, meeting, which was held in Cardiff in conjunction with the South-Western Operational Research Discussion Society. The Annual General Meeting, at which the Chair was taken by Professor M. G. Kendall throughout, due to the unforeseen absence abroad of the Society's new President, Lord Halsbury, was held on 21 October, 1959, and was followed by a paper given by Mr. W. M. Davies (The Steel Company of Wales Ltd.) and Mr. S. L. Cook (Richard Thomas & Baldwins Ltd.) entitled "Automatic Data Collection and Analysis for Works Investigation—two approaches". When it was known that a number of our American colleagues would be visiting London following The Institute of Management Sciences' meeting held in Paris, the opportunity was taken to organize an extra meeting for the Society on 14 September, in conjunction with TIMS: Mr. G Brigham (consultant) spoke on "Towards the Unification of Military and Industrial Operations Research", Dr. M. E. Salveson (President of TIMS) spoke on "Operations Research and Psychoanalysis in the Solution of Management Problems" (this paper had been prepared in conjunction with Dr. H. Holt, Center for Advanced Management, New Canaan) and Professor C. West Churchman (University of California) gave a paper entitled "Ethical Commitments of Operations Research": a dinner followed the meeting.

The rest of the winter programme was as follows:—

1959

27 November		Presidential Address by The Rt. Hon. The Earl of Halsbury, F.R.I.C., F.Inst.P., "The Art of Exposition".
30 December		Operational Research in Advertising.
Part 1	*Speakers:*	Dr. B. Benjamin (General Register Office). Professor W. P. Jolly (Royal Naval College, Greenwich). Mr. J. Maitland (English Universities Press Ltd.).
	Subject:	Theories of Response.
Part 2	*Speakers:*	Mr. A. J. Burkart (British European Airways). Mr. A. M. Lee (British European Airways).
	Subject:	Optimization of Expenditure.

1960

29 January	*Speaker:*	Dr. B. O. Koopman (U.S. Operations Research Groups, London).
	Subject:	Operational Problems with Several Variables to Optimize.
8 March		As mentioned above, this meeting was held in Cardiff.
	Speakers:	Mr. M. G. Bennett. Mr. R. A. Leaver. Mr. H. J. M. Jones.
	Subject:	Operational Research in the British Transport Commission. (*a*) Development of a New Approach to Timetabling Freight Trains. (*b*) Advance Traffic Information at a Marshalling Yard.
24 March	*Speaker:*	Miss B. N. Seear (London School of Economics).
	Subject:	The Growing Contribution of the Human Sciences to Industry.
20 April	*Speaker:*	Dr. K. D. Tocher (The United Steel Companies Ltd.).
	Subject:	An Integrated Project for the Design and Appraisal of Mechanized Decision Making Control Systems.

At the Annual General Meeting the members approved in principle the Executive Committee's proposal to incorporate the Society. At the same meeting Professor Kendall informed the members that as the Journal was now published by Pergamon Press Ltd. the Business Editor of the Journal, Mr. Max Davies, had resigned and Professor Kendall took the opportunity to thank Mr. Davies for the very useful services he had given over eight years: this was warmly endorsed by the meeting. The results of the elections were announced: Lord Halsbury was declared elected as President of the Society for the two years following: Mr. B. H. P. Rivett and Mr. B. D. Hankin were returned unopposed as Honorary Secretary and Honorary Membership Secretary: Mr. E. C. Williams, as mentioned in the previous Annual Report, had resigned as Honorary Treasurer and Mr. G. Norton was declared elected to that post: and Mr. Stafford Beer was declared elected to replace Mr. H. G. Jones, who retired under the four-year rule.

At 31 July, 1960, the membership was: 278 full, 306 associate. (At 31 July, 1959, the figures were 267 full, 257 associate.)

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The Education Sub-Committee continues to keep close links with institutions offering extensive courses in operational research and reports below for information.

A one-day conference on Education in Operational Research was held at Cranfield in July, with limited numbers attending. It is planned to publish a résumé of the Conference in the *Operational Research Quarterly*.

Members of the Society will no doubt be glad to learn that in addition to the well-known Birmingham M.Sc. course in operational research, diploma courses will be available in 1961 at the College of Aeronautics at Cranfield, and Imperial College.

A graduate apprentice training scheme commenced in October, 1960, at the London School of Economics, in conjunction with the British Iron and Steel Research Association. It is understood that L.S.E. may also be offering open courses in the near future in addition to the fortnight's advanced course held at Easter.

Circulation of the *Quarterly* continues to increase, and at 31 July was: United Kingdom 792, United States of America 373, other countries 477; making a total of 1,642.

A scheme has been launched to increase the scope of the abstracts appearing in the *Quarterly* and quite a number of members are assisting: it is planned that this Society will co-operate with other national societies, under the aegis of the International Federation, and contribute to an international journal of abstracts, the plans for which are under way.

No Society national conference was planned for 1959/60 as the second international conference on operational research organized by the International Federation of Operational Research Societies was held from 5–9 September in Aix-en-Provence, the proceedings of which will be published by English Universities Press early in 1961 (American members should approach John Wiley & Sons Inc.).

Sir Charles Goodeve has appointed Miss G. M. Heselton to be Assistant Secretary of the International Federation.

The Executive Committee has been discussing at length the future status of the Society and at a meeting held on 12 July, 1960, they decided by majority vote "that the current status of the Society as a learned Society shall remain unchanged for a period to be agreed": after further discussion the period was unanimously agreed to be 18 months. A note was made at the meeting that the qualification of full member of the Society was not a professional qualification, and meant only that the member was eligible for election to the Executive Committee: these discussions had taken into account the opinions of various members of the Society.

The Committee regret that they are losing the services of Mr. J. Stringer (vice Mr. R. S. Gander), who is due to retire under the four-year rule: they are grateful to him for all the support he has given.

Signed:

HALSBURY (*President*)
M. G. KENDALL (*Vice-President*)
WILLIAM SLATER (*Vice-President*)
S. BEER
S. L. COOK
K. PENNYCUICK
J. STRINGER
B. H. P. RIVETT (*Honorary Secretary*)
G. NORTON (*Honorary Treasurer*)
B. D. HANKIN (*Honorary Membership Secretary*)

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SUB-COMMITTEES OF THE SOCIETY

Membership:	Mr. B. D. Hankin (*Chairman*)
	Dr. T. E. Easterfield
	Mr. H. G. Jones
Editorial:	Mr. R. T. Eddison (*Chairman*)
	Mr. R. H. Collcut
	Mr. S. L. Cook
	Professor P. M. Morse
	Dr. R. J. Smeed
	Mr. H. R. W. Watkins
Education:	Dr. K. Pennycuick (*Chairman*)
	Mr. S. L. Cook
	Dr. F. G. Foster
	Dr. G. A. Garreau
1960/61 *Programme:*	Mr. B. H. P. Rivett (*Chairman*)
	Mr. R. T. Eddison
	Mr. G. W. Sears
	Mr. S. Beer

FINANCIAL SUPPORTERS OF THE SOCIETY 1959/60

British Iron and Steel Research Association.
British Petroleum Co. Ltd.
Courtaulds Ltd.
Distillers Co. Ltd.
Electric & Musical Industries Ltd.
Ferranti Ltd.
Arthur Guinness Son & Co. (Park Royal) Ltd.
International Computers & Tabulators Ltd.
Pilkington Bros. Ltd.

Rolls-Royce Ltd.
Shell-Mex & B.P. Ltd.
Shell Petroleum Co. Ltd.
Urwick Orr & Partners Ltd.

OPERATIONAL RESEARCH SOCIETY
1960 ANNUAL GENERAL MEETING OF THE SOCIETY

THE following are the Minutes of the Annual General Meeting of the Society, held on 26 October, 1960, at 6 p.m. at the Royal Society of Arts, 6 John Adam Street, London, W.C.2: they are subject to confirmation at the 1961 Annual General Meeting.

The Chair was taken by the President, Lord Halsbury.

(1) The Minutes of the Annual General Meeting held on 21 October, 1959, were confirmed and signed.

(2) The Executive Committee's report for 1959/60 which had been previously circulated, was introduced by the President. The Chairman moved the adoption of the report: motion passed.

(3) The Honorary Treasurer, Mr. G. Norton, said that the accounts which had been circulated with the Executive Committee's report were being revised by the auditors. Duplicated copies of the revised accounts were available at the meeting. The Honorary Treasurer moved the adoption of the revised accounts subject to the completion of the audit: motion passed.

(4) The President made the following announcements regarding the nominations of the officers:

Honorary Secretary: Mr. B. H. P. Rivett had been returned unopposed.

Honorary Treasurer: Mr. G. Norton had been returned unopposed.

Honorary Membership Secretary: Mr. B. D. Hankin had been returned unopposed.

Executive Committee Member: Mr. H. R. W. Watkins.

The motion was put to the meeting that the above be and are hereby elected: motion passed.

(5) Election of auditors. Messrs. Knox, Cropper, Gedge & Co. were re-elected auditors for the forthcoming financial year.

(6) Any other business. The Secretary read out a letter which she had received from one of the members of the Society. The points he raised were noted by the Honorary Secretary.

There being no other business the Annual General Meeting was concluded at 6.25 p.m.

EXTRAORDINARY GENERAL MEETING OF THE SOCIETY

An Extraordinary General Meeting of the Operational Research Society was held on 26 October, 1960, at 6.30 p.m., at the Royal Society of Arts, 6 John Adam Street, London, W.C.2. The Chair was taken by the President, Lord Halsbury.

After an introduction by the President and by Mr. J. H. Thomson of the Society's solicitors, Messrs. Allen & Overy, the following resolution was passed:

> "That the affairs of the Society be and they are hereby wound up, that the assets and liabilities of the Society be transferred forthwith to a Company limited by Guarantee known as the Operational Research Society Limited formed to continue the activities of the Society, and that the Executive Committee be and they are hereby authorized to take such steps as may be necessary to complete such winding up and transfer as aforesaid."

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NEWS OF MEMBERS

New Full Members of the Society

Mr. B. R. Aston, International Computers & Tabulators Ltd., London.
Dr. J. Y. Barry, Arthur Andersen & Co., New York.
Mr. R. Beresford, The War Office, Forthalstead.
Dr. E. L. Brink, University of Pennsylvania.
Mr. R. R. Crane, Touche, Ross, Bailey & Smart, New York.
Mr. R. B. Davies, Central Electricity Generating Board, Cardiff.
Mr. M. A. Dizy, Richard Thomas & Baldwins Ltd., Newport.
Mr. D. J. Finch, British Transport Commission, London.
Mr. J. S. James, Albert E. Reed & Co. Ltd., near Maidstone.
Mr. L. S. Jay, County Planning Officer, Lewes.
Mr. P. J. Judge, T.I. Technological Centre, Walsall.
Mr. D. J. Kirby, National Coal Board, London.
Dr. S. Laner, BISRA, London.
Dr. R. F. Layton, U.S. Operations Research Groups, London.
Mr. A. J. Mayne, Leeds University.
Dr. N. Morris, Richard Thomas & Baldwins Ltd., Ebbw Vale.
Mr. H. V. Murphy, United Kingdom Atomic Energy Authority, Warrington.
Dr. E. Page, Steel Company of Wales Ltd., Port Talbot.
Mr. W. E. Prior, GKN Group Research Laboratories, Wolverhampton.
Professor R. W. Revans, Manchester College of Science and Technology.
Mr. G. G. Stephenson, National Coal Board, London.

Changes of Appointment

Mr. J. Banbury—previously with ORbit (Operational Analysis & Research) Ltd., is now with Richard Thomas & Baldwins Ltd., Newport.

Dr. D. J. Bartholomew—previously at the University College of North Staffordshire, is now with the University College of Wales, Aberystwyth.

Mr. J. C. Beresford—previously with the National Coal Board, is now at the Manchester College of Science and Technology.

Mr. J. J. Bissell Thomas—previously with Morgan Crucible Co. Ltd., is now with Saxone, Lilley & Skinner Holdings Ltd., Kilmarnock.

Dr. J. Boyes—previously at the Royal College of Science and Technology, is now at Ashridge College, Berkhamsted.

Dr. W. R. Buckland—resigned from the London Transport Executive to take charge of the new statistical advisory service of The Economist Intelligence Unit, London.

Mr. W. L. Creswell—previously with Short Bros. & Harland Ltd., is now with Ferranti Ltd., Manchester.

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Mr. M. A. Dizy—previously with A.E.I. (Manchester) Ltd., is now with Richard Thomas & Baldwins Ltd., Newport.

Mr. P. G. Lucas—previously with The United Steel Companies Ltd., is now with Albert E. Reed & Co. Ltd., near Maidstone.

Dr. P. B. Myers—previously with the Bell Telephone Laboratories, is now with Motorola, Inc., Phoenix, Arizona.

Mr. M. J. H. Nightingale—previously with McKinsey & Co. in London, is now with McKinsey & Co. in New York.

Mr. A. C. Page—previously at the Medway College of Technology, is now with Richard Thomas & Baldwins Ltd., Scunthorpe.

Mr. E. D. van Rest—previously with the Ministry of Supply, has been appointed lecturer in quantitative methods of operational research at Cambridge University.

Mr. C. A. Warner—previously with the US Operations Research Group, London, is now with The Johns Hopkins University, Bethesda, Maryland.

Society 1961 *National Conference*

The 1961 Conference will be held from 3 to 5 October at the Grosvenor Hall Conference Centre, Ashford, Kent. Members will be sent details as soon as they are available.

SOUTH-WESTERN OPERATIONAL RESEARCH DISCUSSION SOCIETY

Notes of a Meeting held in Cardiff, 11 *January*, 1961. *Paper by Dr. M. G. Simpson* (*Messrs. Touche, Ross, Bailey & Smart*), "*Dynamic Programming*".

Dr. Simpson dealt with the concept behind the technique of Dynamic Programming outlining Bellman's Principle of Optimality. He dealt with a simple transportation problem and went on to instance the sort of problems in multi-stage decision processes which Dynamic Programming might help to solve.

Dr. Simpson went into considerable detail to describe the method of computation and stressed that it was essentially a way of treating a multivariate problem in terms of a multiplicity of single variates. He continued with some examples of an investment and an auction bidding problem.

In the discussion which followed Mr. R. Acton (R.T.B.) asked why Dynamic Programming had received such wide publicity without having had much practical pay-off. Dr. Simpson agreed that he was not able to give any industrial applications, but that this was usual with a developing technique.

Mr. J. K. Friend (B.R.P.) instanced an aircraft refuelling problem which might have been amenable to solution by D.P. methods.

Mr. D. Robinson (R.T.B.) described a problem which had recently been solved using Dynamic Programming concerning the stocking of sheet steel in different weights and widths: the problem being to decide which weights should be stocked in order to satisfy orders with the minimum give-away of weights.

The Chairman, Mr. G. W. Thomas (S.C.O.W.) thanked Dr. Simpson for his paper and suggested that Mr. David Robinson had talked himself into a contribution at a later date.

The total attendance at the meeting was 43.

R. B. D.

ANNUAL CONFERENCE OF AKOR (ARBEITSKREIS OPERATIONAL RESEARCH) MÜNSTER (WESTPHALIA) 6–8 OCTOBER, 1960

The Annual Conference of AKOR, the German association of people actually earning their living by operational research in industry, together with some of their academic friends, attracted some 80–90 participants. A wide range of German industry was represented, and there were members and visitors from Switzerland, Austria, France and Great Britain.

The programme which, as always, was too full, contained a thoroughly representative cross-section of operational research interests. There was an eloquent plea that experiments can still profit from being designed in the classical statistical manner, from Professor Dr. Fr. Schwerdtfeger of Hanover. Professor Dr. H. P. Künzi, of Zurich, gave a distinguished survey of the present state of non-linear programming, and some recent good ideas relating to linear programmes on computers were discussed in other sessions. There were simple introductions to queueing theory, computers, traffic problems, and the practical use of matrices in works records. Simulation was represented by the simulation of a French motor bus route, and tests of a large variety of possible priority rules for programming an engineering job shop, carried out by simulation at Cornell.

A mathematical method of forecasting a rate of economic growth was expounded in a lively session by Dr. E. Nievergelt of Berne, and a method for assisting with the econometric decisions required of a government body was described by Dr. H. Gülicher, of Münster University.

One excellent piece of industrial operational research was described by Dipl.-Math. W. Schulz of Elberfeld, who had dealt successfully with the problem of deciding whether to change from returnable to expendable packaging material, in circumstances where data were particularly difficult to obtain.

In the final session Dr. J. J. Weidmann of Nestlé (Vevey) and Mr. P. G. Lucas, then of United Steels, Sheffield, both tried to describe what it is like actually to use operational research in a large industrial firm.

All the stock characters of operational research were present: the rare operational research scientist who can report original solutions to a tricky practical problem; the young statisticians who enthusiastically describe Monte Carlo methods and queues that could never exist; the old hands who try to bring the discussions down to earth; the experienced mathematicians who survey the methods coolly; the young mathematicians who gleefully relate how their series was made to converge after unanticipated excursions to infinity; and the people who just sit there and don't say anything.

The problems of German operational research were discussed in many informal conversations. They appear to be very similar to those of operational research in the rest of the world—recruitment, training, the conduct of operational research under a management which may not understand its needs, the defence of operational research as an industrial function against those who claim sole rights over it as an academic domain. The quality of papers was naturally mixed, with a higher than world average proportion of worthwhile contributions.

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Feasibility and Financial Studies of a Port Installation*

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The Steel Company of Wales Limited

The subject of this paper is a computer simulation of the operations of an iron ore unloading port. The object of the work was first, to estimate the advantages of reducing some of the delays caused by the port's physical limitations, and second, to estimate the capacity of the existing system and determine whether this could be modified to handle estimated future foreign ore requirements.

Part 1 provides a brief background to the project, including its evolution from earlier work, and a description of the shipping operations and plant with which the work is concerned.

The formulation of the model, including the simulation of scheduled and random arrivals, is described in Part 2. This is followed in Part 3 by an account of the programming details, the logic of the unit scan procedure, and the arrangement of results.

The choice of dependent variables and the treatment of results are discussed in Part 4, which also includes a summary of the conclusions drawn from the results and an indication of the use made by management of them.

The authors' company requires to import increasing quantities of foreign iron ore from increasingly distant ports. The problems arising concern the economic aspects of shipping operations, the feasibility of using other ports as transhipment terminals, and the future of the company's own port. This paper described work relating to this third aspect only.

PART 1: INTRODUCTION

Background

THE initial object of the work was to estimate the financial and operational advantages to the Company of deepening the entrance channel to the dock, thereby reducing its physical limitations and the incidence of tidal delays. Having formulated the model, the object was extended to estimating the capacity of the system under current conditions, and to determining what increase in capacity could be expected from relatively inexpensive modifications incorporating the deeper entrance channel and estimates of reduced service times derived from a method study of unloading operations at the wharf.

In the first instance, attempts were made to obtain a mathematical solution. Several models were formulated but failed to reproduce current or future conditions accurately enough when tested against wharf records and the results of hand simulations. The reason for this failure was that a mathematical

* Paper given before the Operational Research Society on 27 March, 1961.

formulation could not adequately incorporate the heterogeneity of ships visiting the wharf, the nature of their arrivals or the bulk-service operation of the lock. For this reason, it was decided to use the Monte Carlo simulation technique. These factors are discussed in more detail later in the paper.

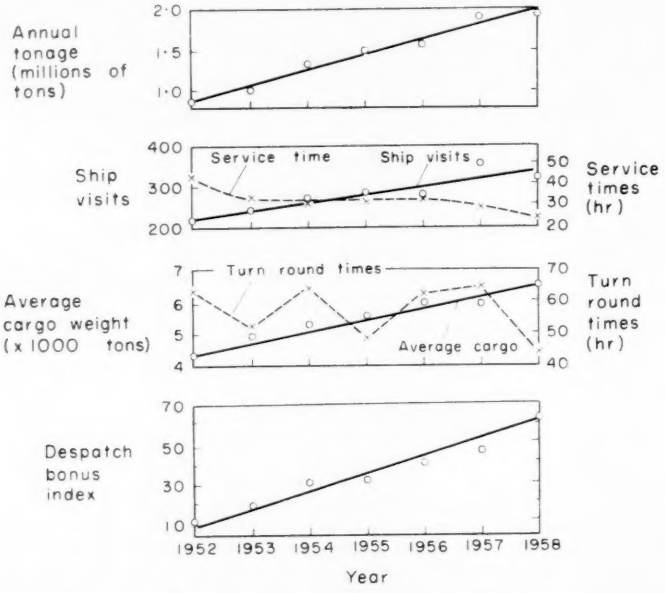

FIG. 1. *Changes in operating conditions from* 1952 *to* 1958.

Several estimates of the port's capacity had been made by manual simulations in the preceding seven years, but each in its turn was subsequently vitiated by changes in operating conditions, the extent of which can be seen in Figure 1. To avoid a team of people being employed for long periods on updating estimates of the port's capacity, it was decided to prepare a simulation program for a Ferranti Pegasus Mark I Computer such that actual or predicted changes in the system could be incorporated with the minimum of program modification.

Description of Plant

The plant with which this work is concerned is an iron ore unloading wharf in South Wales.

As illustrated in Figure 2, ships arriving in the bay proceed to the wharf via a 1000 yard long tidal entrance channel, and a lock through which ships carrying a maximum of approximately 9000 tons of ore can enter the dock.

Between arriving in the bay and tying up at the wharf, ships are subject to two types of delay. The first of these, neaping, is governed by the tidal characteristics[1] and the physical limitations of the entrance channel, and occurs when

there is insufficient water in the entrance channel at high tide to allow entry to the lock. While the datum level for published tide heights is the level of the outer lock-cill, effective tide heights are reduced because the entrance channel is dredged to three feet above tide datum level. The extent to which neaping affects port operations can be judged by the fact that it limits the entry of ships carrying 9000 tons of ore to approximately 60 per cent of the tides. The second type of delay, queueing, is the result of congestion in the system, and occurs either when a ship has to wait to use the lock even though it is not neaped, or when it has to wait for a berth at the wharf. These delays, plus operational times such as unloading and travelling to and from the wharf, constitute a ship's turn-round time, i.e. the time from its arrival in the bay to its passage out through the lock.

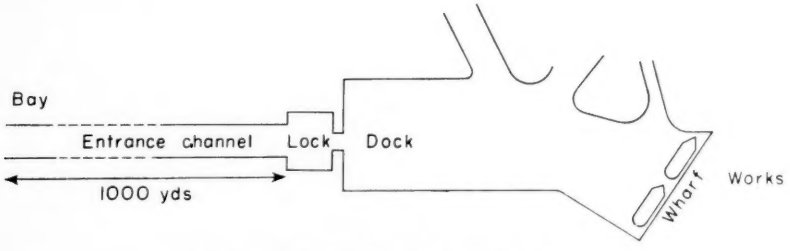

FIG. 2. *Diagrammatic layout of the port.*

The wharf itself is a two-berth installation comprising five transporter-type cranes capable of discharging to a stockyard, on to belts to an ore preparation plant, or direct to blast furnace bunkers.

Iron ore is purchased through a central buying agency which operates a despatch bonus scheme whereby importers receive a bonus for the fast turn round of ships. The bonus per ship varies from 5 units per ton for a rate of 8000 tons a day or higher to 1 unit for an unloading rate of 2000 tons a day, and nothing for lower rates.

PART 2: DESCRIPTION OF MODEL

Number of Ship Visits per annum

The first step in defining the model was to determine the number of ship visits required for the increasing annual tonnages to be considered. Three types of ship visit the wharf:—

(i) ore carriers, designed for this port, with a cargo weight of about 9000 tons,
(ii) 'K' ships, carrying Swedish ores, with a cargo weight of about 5500 tons,
(iii) general cargo ships, with cargo weights of between 2500 and 8500 tons.

Forecasts of the annual requirements of the various types of ore were supplied by management. These were translated into numbers of visits of the three classes of ship referred to above, by allocating to them, subject to various known and

estimated restrictions on ship availabilities, the same proportion of each type of ore as they discharged at the wharf in 1958. The results of one such calculation are shown in Table 1.

TABLE 1. AN ESTIMATE OF FUTURE ANNUAL ORE REQUIREMENTS AND SHIP VISITS (Tons × 1000)

Ores	Total tonnage	Ore carrier		'K' class		General cargo	
		Tonnage	No. of visits	Tonnage	No. of visits	Tonnage	No. of visits
Kiruna	811	554	63	232	40	25	3
Granges/Gelli	481	361	40	90	16	30	5
Purchased fines	582	119	14			463	107
Softs	429	278	33			151	29
Concentrates	728	592	67			136	18
Labrador	460	460	53				
Total	3491	2364	270	322	56	805	162

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Arrival Patterns

Distributions of ore carrier inter-arrival times and arrivals per unit time (as distinct from 'K' ships and general cargo ships) were drawn up from an analysis of wharf records. These were compared with the negative exponential and Poissonian distributions respectively using χ^2, and gave non-significant differences. This indication of apparent random arrivals was then tested on "dry runs", the results of which showed beyond doubt that ore carriers do not arrive at random. The neaping delays of the "dry runs" with random arrivals were very significantly greater than those for the corresponding actual periods, which led to the conclusion that ore carriers arrive in a pattern which tends to reduce neaping.

This experience illustrates both the lack of power of the χ^2 test and the importance of considering errors of the second kind.

A more detailed analysis of 1958 wharf records showed that the probability of an ore carrier arrival on any tide depends on the predicted height of that tide, and its position in the tide cycle. A typical cycle of high tides is shown in Figure 3.

The horizontal line drawn at 31 ft represents the tide height required for an ore carrier to enter the dock.

In the interval AB, when there is little chance of neaping, arrivals are very frequent; from B they become less frequent until in the interval CD, they are practically negligible apart from those on the first two or three tides after C, due to ships arriving later than expected. In the interval DA, the frequency of arrivals increases as tide heights approach the 31 ft required to enter dock.

To express this quantitatively, the tides were divided into groups of one foot, those in the interval BCD being given a negative sign and the remainder a positive sign. The basic mean arrival rates per tide for these groups were then

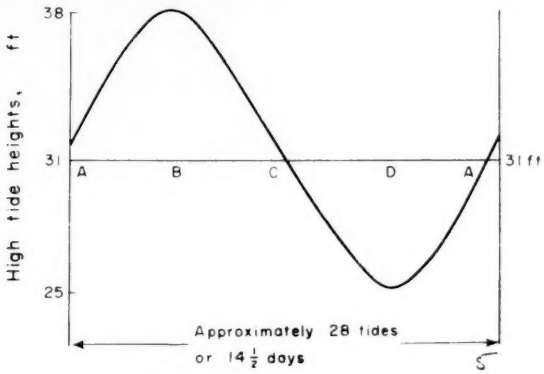

FIG. 3. *Representation of a high tide cycle.*

obtained by dividing the number of arrivals in each group in 1958 by the number of tides in the respective groups in the same year. The results of this calculation are shown in Table 2.

To obtain the corresponding mean arrival rates for the conditions to be simulated, the rates shown in column 4 of Table 2 were multiplied by

$$\frac{\text{required number of arrivals}}{\text{1958 arrivals}}.$$

The distributions associated with the mean arrival rates of the above groups were assumed to be Poissonian; knowing the group to which a tide belonged, the number of arrivals on that tide could then be sampled from the appropriate distribution.

Ore carriers carrying Labrador ore had to be treated separately because the loading port is icebound during the winter.* To reproduce the seasonal effect, the above procedure was adopted for summer tides, the probability of an arrival on winter tides being fixed at zero.

Arrivals of 'K' and general cargo ships showed no detectable scheduling. The arrival pattern of these ships was, therefore, taken to be random throughout the year, arrivals per tide being sampled from a Poisson distribution with mean

$$= \frac{\text{no. of arrivals/year}}{\text{no. of tides/year}}.$$

* The annual capacity of the port is partly dependent on the proportion of seasonal ore imported. Whilst it is not discussed in this paper, the relationship was studied and its effect incorporated in the final estimates.

TABLE 2. ORE CARRIER MEAN ARRIVAL RATES FOR THE YEAR 1958

Tide range (ft)	No. of tides	No. of arrivals	Mean arrival rate per tide
A			
+31·1–32·0	35	4	0·11429
+32·1–33·0	60	11	0·18333
+33·1–34·0	47	16	0·34043
+34·1–35·0	24	3	0·12500
+35·1–36·0	25	3	0·12000
+36·1–37·0	17	2	0·11765
+37·1–38·0	19	4	0·21053
+38·1–39·0	2	0	0
B			
−38·1–39·0	0	0	0
−37·1–38·0	8	2	0·25000
−36·1–37·0	14	4	0·28571
−35·1–36·0	12	2	0·16667
−34·1–35·0	19	4	0·21053
−33·1–34·0	32	5	0·15625
−32·1–33·0	41	12	0·29268
−31·1–32·0	34	12	0·35294
C			
−30·1–31·0	28	6	0·21429
−29·1–30·0	37	1	0·02703
−28·1–29·0	34	1	0·02941
−27·1–28·0	36	0	0
−26·1–27·0	31	0	0
−25·1–26·0	10	1	0·10000
D			
+25·1–26·0	5	1	0·20000
+26·1–27·0	16	0	0
+27·1–28·0	29	4	0·13793
+28·1–29·0	31	2	0·06452
+29·1–30·0	30	6	0·20000
+30·1–31·0	30	3	0·10000

A B C D correspond to the points shown on Figure 3.

Delays Outside Lock

Because of the gradual ebb and flow of the tide there is a limit to the time about any high tide that the lock can be worked. Figure 4 shows the timing of this ebb and flow for various high tide levels.

It can be seen from this that a ship requiring 31 ft of water can enter the lock on a 36 ft high tide from 2 hours before to about $1\frac{3}{4}$ hours after the high tide time. To facilitate the incorporation of these locking restrictions into the model, the rules shown in Table 3 were established from Figure 4.

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The determination of ore carrier and 'K' ship neaping delays was straightforward because they are both quite uniform classes of ships and could, therefore, be considered as having constant draughts. Since it is the draught of a ship that determines the tide height required for it to enter the lock, the relevant height for these types of ship could be taken as constant—31 ft for ore carriers and 27·1 ft for 'K' ships.

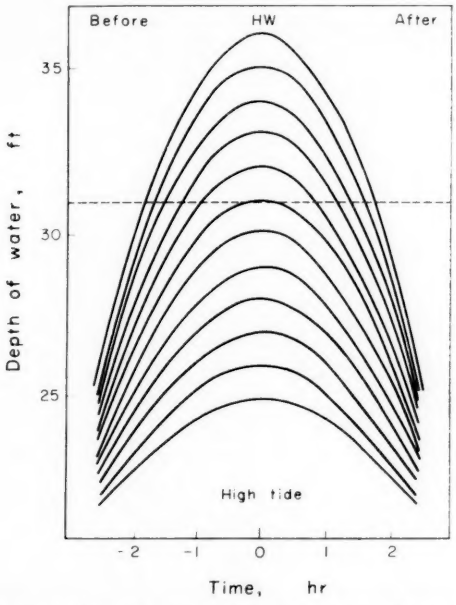

FIG. 4. *Variation in depth of water at lock for 2 hours either side of high tide.*

TABLE 3. NUMBER AND TYPE OF SHIPS ABLE TO PASS THROUGH LOCK ON GIVEN HIGH TIDE HEIGHTS

Tide Ht. (ft)	Maximum number of ships able to pass through lock
<29	(2 others)
$\geqslant 29$ $\leqslant 32$	(2 others) or (1 O.C. + 1 other)
>32 $\leqslant 34$	(3 others) or (1 O.C. + 2 others) or (2 O.C. + 1 other)
>34	(4 others) or (1 O.C. + 3 others) or (2 O.C. + 2 others)

O.C. = Ore carrier; 'Other' = 'K' ship or general cargo

General cargo ships, on the other hand, do not exhibit this homogeneity. As mentioned earlier, their cargo weights vary from 2500 tons to 8500 tons and an analysis of wharf records revealed a strong relationship between their cargo weights and draughts, illustrated in Figure 5.

This relationship was used to determine the tide height required for entry, by first sampling a cargo weight (x) in tons and determining the corresponding draught (y) in feet from the equation $y = 0{\cdot}0013x + 14{\cdot}87$. The value of y so obtained was then modified by the addition of a sampled draught deviate. The tide height required for entry was found by adding to the draught the customary allowance for bottom clearance and an allowance for the difference between tide datum level, i.e. the level of the outer lock-cill, and the dredged channel level of the condition being simulated.

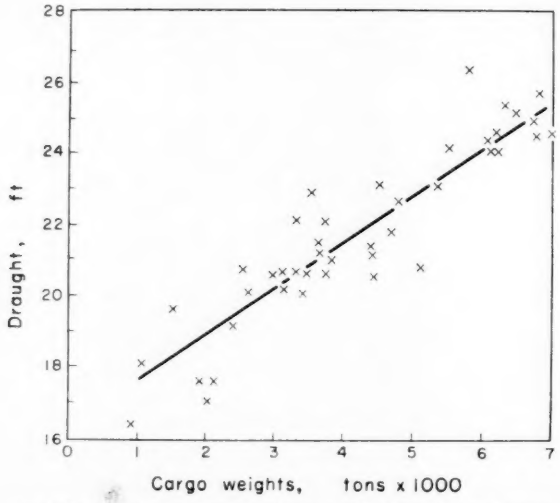

FIG. 5. *Relationship between general cargo ship draughts and cargo weights.*

Service Times

Ore carrier and 'K' ship berth times were determined by a random sample from very nearly normal distributions drawn up from an analysis of records. In the case of general cargo ships, however, cargo weights had at this stage been sampled to determine draughts, and these were used to estimate berth times from the relationship shown in Figure 6.

It will be seen that Figure 6 does not have a homogeneous residual about the line; this was overcome by making the usual logarithmic transformation. Individual values of the berth times were obtained from the equation $y = {\cdot}0649x^{{\cdot}6804}\, e^{{\cdot}2513r}$ where x is the sampled cargo weight and r is sampled from a normal distribution of zero mean and standard deviation 1.

This equation was used only when considering conditions involving current service times. A method study of unloading operations was undertaken by the Operational Research Department to estimate what reduction in service times could be expected from modifications to the existing installation and practice.

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When considering conditions involving reduced service times, the relationships used were obtained by amending the above equation to reproduce unloading rates shown possible by the results of the method study.

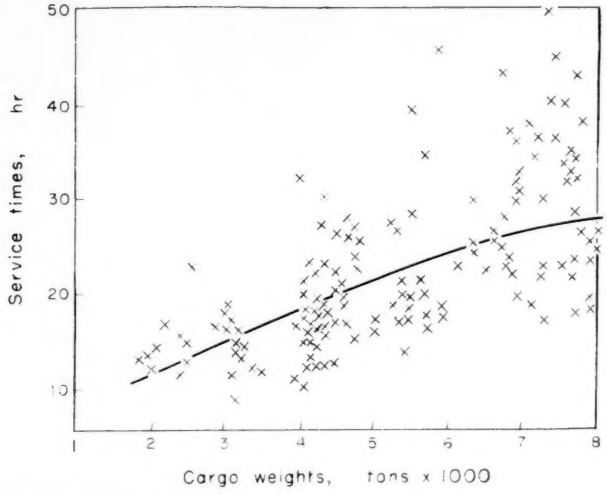

FIG. 6. *Relationship between general cargo ship service times and cargo weights.*

Despatch Bonus

The rules for despatch bonus earnings allow time to become effective either from the commencement of discharging, or from 8.00 a.m. on the day after arrival, whichever is the earlier.

This meant, that besides knowing the time of the tide on which a ship arrives, it was also necessary to determine the actual time of arrival in the bay. This was done by sampling a deviate to subtract from the tide time.

PART 3: COMPUTER PROGRAMS

Arrivals

Figure 7 shows a simplified flow diagram of the program. It will be seen that the arrival pattern was pre-determined and incorporated on an arrivals tape containing:

(i) consecutive tide number
(ii) time between tides
(iii) published tide heights
(iv) ship arrivals.

This was done because it was estimated that the program for the generation of arrivals would take as long to develop as that for the remainder of the model, and splitting the model in this way enabled the development of both parts to proceed concurrently under different programmers.

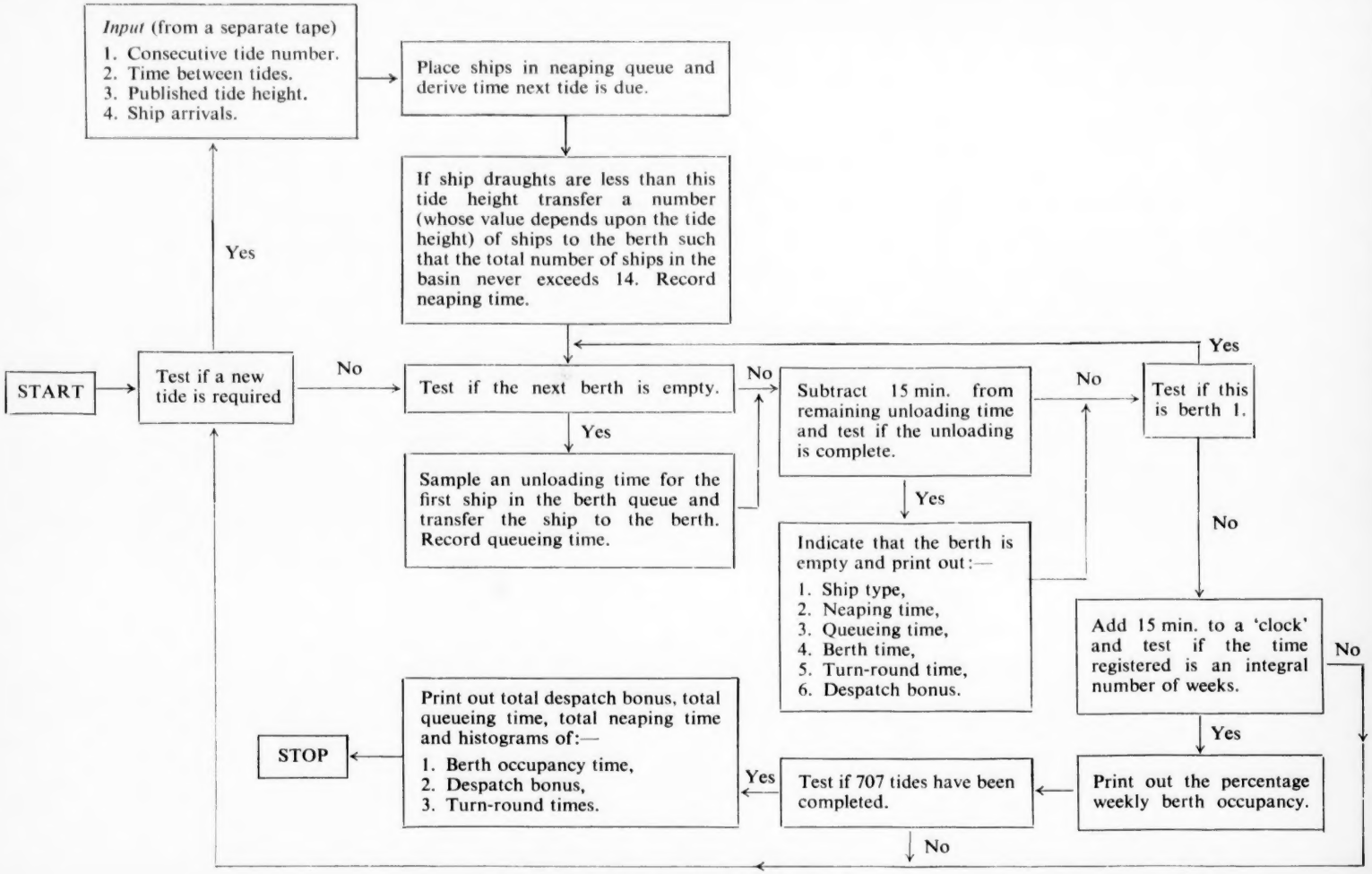
Input (from a separate tape)
1. Consecutive tide number.
2. Time between tides.
3. Published tide height.
4. Ship arrivals.
Place ships in neaping queue and derive time next tide is due.
If ship draughts are less than this tide height transfer a number (whose value depends upon the tide height) of ships to the berth such that the total number of ships in the basin never exceeds 14. Record neaping time.
START
Test if a new tide is required
Yes
No
Test if the next berth is empty.
No
Yes
Sample an unloading time for the first ship in the berth queue and transfer the ship to the berth. Record queueing time.
Subtract 15 min. from remaining unloading time and test if the unloading is complete.
No
Yes
Test if this is berth 1.
Yes
No
Indicate that the berth is empty and print out:—
1. Ship type,
2. Neaping time,
3. Queueing time,
4. Berth time,
5. Turn-round time,
6. Despatch bonus.
Add 15 min. to a 'clock' and test if the time registered is an integral number of weeks.
No
Yes
Print out the percentage weekly berth occupancy.
Test if 707 tides have been completed.
Yes
No
Print out total despatch bonus, total queueing time, total neaping time and histograms of:—
1. Berth occupancy time,
2. Despatch bonus,
3. Turn-round times.
STOP

FIG. 7. *Flow diagram of the port simulation.*

Details of the basic procedure for generating arrivals have already been given. To effect the random sampling involved—and this applies to all the sampling required in the simulation—it was more convenient from the programming point of view to generate pseudo random numbers internally by the Multiplicative Congruence Method[2,3] due to Lehmer, rather than store random numbers in the machine.

In practice a type of feedback applies between the Company and the buying agency. The Company, having placed an order for a year's supply of ore, tries to adjust supplies in accordance with monthly stocks. An attempt was made to incorporate such a feedback into the model, by comparing at the end of each month actual arrivals with expected arrivals, and altering accordingly the probabilities of arrivals for the subsequent month's tides. This did not prove at all successful, as it was found that nearly all the correction came in the last month, and for lack of time the idea could not be developed.

Allowing the above sampling procedure free play, however, led to such a degree of variation in the number of arrivals per year generated for a given condition, that to incorporate these raw arrival patterns into the model would have entailed the simulation of something like 20 to 30 years' operation for each scheme examined to reduce the error of the results to reasonable bounds. In one case, for example, for an expected 217 visits of one type of ship, the number actually generated varied from 200 to 230.

After successful tests with actual data, the following method was adopted to control within closer limits the number of arrivals on the tapes used in the simulations. Having chosen a starting value, 706 pseudo random numbers were generated, 706 being the average number of tides in a year. The mean of those numbers was then calculated and the starting value rejected if that mean was more than one standard error away from the expected mean. The next step was to reject those starting values that, although passing the first test, resulted in a number of arrivals outside ± 5 of the required number.

This procedure was adopted for each of the three types of ship, one pattern for each being combined to form one arrivals tape.

Organization of Complete Program

At the commencement of a run on the computer, the program, relationships and distributions were fed into the machine and stored on the drum. Upon initiating the program, the data for 4 tides contained on the arrival tape were read in and processed. This takes about 43 seconds and was repeated until the data for the 706 tides had been processed.

The amount of storage taken up by data, program, etc. can be seen from Table 4.

The model described in Part 2 is such that, had it been the only consideration, it would have been programmed to proceed on an event-to-event basis. At the time, however, it was thought that there might be need to extend the model to

beyond the stage of unloading, in which case the interactions that would have to be incorporated would render the formulation of the logic of the event-to-event procedure an intractable problem. It was, therefore, decided to program the simulation on a unit scan basis.

TABLE 4. CONTENTS OF PEGASUS MAIN STORE

Contents	No. of blocks
Program	108
Sub-routines	20
Distributions	82
Working space	109
Results	129

The program used a fixed time base and required time to be registered on three clocks:—

(i) the basic clock

(ii) the tide clock, i.e. a record of the time of the next high tide

(iii) the weekly clock, i.e. a record of the time corresponding to the end of the current week.

Whenever the two berths had been examined and the subsequent activities completed, the basic clock was advanced by 15 minutes, i.e. the basic scan time.

On arrival, ships were placed in the neaping queue in the following order of priority:—

ore carriers, 'K' ships and general cargo ships.

Each ship was given an identification number and a queue entry time, i.e. the time registered by the basic clock on joining the queue.

At every high tide the neaping queue was examined and a number of ships removed from it and placed in the berth queue according to the rules of Table 3. The neaping queue entry time of each ship transferred was subtracted from the present basic clock time and the result stored as the neaping delay for that ship. Present basic clock time was then used as the berth queue entry time.

As soon as a berth became available the first ship in the berth queue was transferred to that berth, and its queueing time, obtained by subtracting the berth queue entry time from the present basic clock time, was stored. The berth time, determined as described in Part 2, was then transferred to the berth register.

At the end of each basic scan time, the remaining unloading time of any ship in a berth was reduced by 15 minutes. When the result of this subtraction was zero, the time between the end of unloading and the next tide was calculated to determine whether the ship could sail on that tide. The turn-round time was then calculated, the berth emptied and the output data punched out.

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Table 5 is an illustration of the information printed out for each ship when it had been unloaded. The weekly berth occupancy figure was printed out at the end of each week.

TABLE 5. ILLUSTRATION OF DETAILED PRINT-OUT

Weekly berth occupancy	Type of ship*	Arrival tide	Hours neaped	Hours queued	Berth hours	Turn-round time (tides)	Cargo weight (tons)	Despatch bonus £ s. d.
	3	65	0	0	22·0	2	4400	
	O	68	0	0	27·0	3	8700	
	3	69	0	0	18·4	2	5500	
	2	74	0	0	21·0	2	5400	
	O	75	0	0	27·0	3	8700	
	O	75	0	8·4	39·0	5	8700	
	O	76	0	14·1	21·0	4	8700	
51·5%								

* O = Ore carrier; 2 = 'K' ship; 3 = General cargo ship

The following data were printed out at the end of each year:—

(i) ship hours lost through neaping
(ii) number of ships neaped
(iii) ship hours lost through queueing inside and outside the dock
(iv) number of ships having to queue
(v) total despatch bonus earned
(vi) histograms of turn-round times, service times and despatch bonus earned per ship for each type of ship
(vii) tonnage of ore imported, classified by type of ship and type of ore.

It will be seen that, in addition to a straightforward listing of results, the program catered for a considerable degree of internal analysis, and the extent to which this speeded up the project can be judged by the fact that it involved less than 1 per cent of total machine time, whereas to do the same work outside the machine would itself have taken several man-weeks.

General Points

The print-out shown in Table 5 may seem too detailed, and in fact it was intended initially to use this detail only to assist in proving the efficacy of the program. When, however, the project was extended to a study of port capacity, such a print-out was essential for an examination of berth queue behaviour.

Programming commenced in April, 1959, and was completed in about three months, including proving time. It is felt that the close co-operation maintained between programmers and operational research workers added considerable impetus to the work. The required results were certainly produced much sooner

than would have been possible had the programmers been isolated from the development of the model, and the operational research workers unaware of programming difficulties.

The 1 : 4200 ratio of actual time to simulated time achieved with this program, compared with approximately 1 : 35 for the early hand simulations, is itself sufficient justification of the decision to use the computer, apart from the fact that the program could, without much difficulty, be made to cater for future changes in the parameters of the model.

PART 4: RESULTS

There was no difficulty about the choice of dependent variables to estimate the financial advantages to the Company of deepening the entrance channel. The only source of direct financial gain would be from increased despatch bonus earnings due to reduced turn-round times resulting from reduced neaping and queueing delays.

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TABLE 6. DESPATCH BONUS EARNED (INDEX)

	Extra dredging (ft)		
Run	0	1	2
1	74	87	90
2	63	81	86
3	74	88	96
4	80	94	100
5	66	88	91
6	68	81	93
Mean	71	87	93

Table 6 gives an indication of the nature of the results obtained in estimating the financial advantage, in the form of increased despatch bonus earnings, of deepening the entrance channel.

Criteria for estimating capacity, however, were not nearly so straightforward because there is no clear-cut division between practicable and impracticable throughputs. All that could be done was, with management's co-operation, to determine for each scheme under consideration the range of annual imports within which congestion would gradually become prohibitive. This was done by examining the following dependent variables.

(*a*) *Berth Occupancy and Queueing*

In general, an annual berth occupancy of approximately 70 per cent, together with the occurrence of many consecutive weekly occupancies of over 50 per cent

accompanied by heavy and protracted queueing conditions, was taken to mark the practical limit.

(*b*) *Despatch Bonus Earnings*

The tonnage that maximized the Company's annual despatch bonus earnings was another indication of the practical capacity under given conditions, because the behaviour of the bonus follows the economist's Law of Diminishing Returns. While this measure was not independent of berth occupancy and queueing, it did serve to narrow the limits of the estimates of capacity which were, in general, quoted to ± 2 per cent.

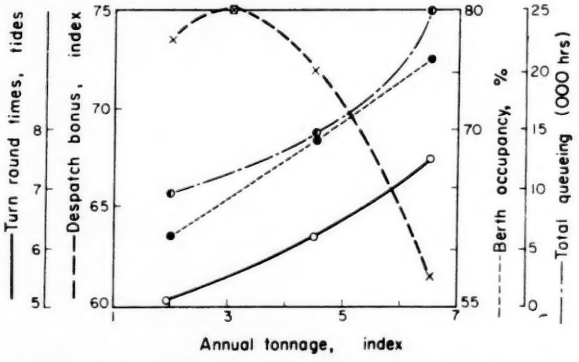

FIG. 8. *Summary of port performance with various throughputs under given conditions.*

As mentioned in Section 3, a lot of analysis was done internally; this included the calculation of the annual despatch bonus earned, berth occupancy, average turn-round times and the total ship time lost by queueing. Figure 8 shows how this summarized information was presented to provide a convenient review of the relevant details of any particular scheme.

The advantages of the detailed print-outs in comparing the degree of congestion associated with the various conditions and throughput can be seen in Figure 9. The heaviest congestion found in each run was plotted in this way, so that to a large extent the experience of management could be used to decide between the practicable and the impracticable. In the two cases illustrated in Figure 9, for example, they were quite prepared to accept the degree of congestion shown in the second, but considered that that shown in the first to be evidence of severe overloading.

The results obtained from the simulations showed that the Company would gain significant financial and operational benefits by deepening the entrance channel, and that the present system could not readily be modified to make the handling of required future tonnages a practicable proposition.

The use made by management of these results can best be judged by the fact that the deepening of the entrance channel has been authorized as a short-term expedient and, as a long-term measure, a major port development programme is being undertaken.

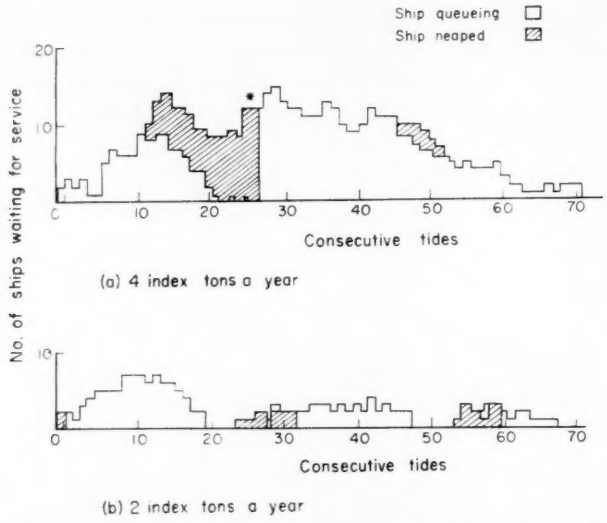

FIG. 9. *Comparison of the worst congestion experienced in the simulation of two annual throughputs of ore, other conditions being constant.*

* This does not imply that all the ships concerned entered the dock on the same tide. From the definitions of neaping and queueing given earlier, it follows that some of the ships will be queueing outside the lock.

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An Appraisal of some of the Problems of Measurement in Operational Research*

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Operational research is invariably concerned with the examination of systems which include human beings. Since the degree of confidence in a decision reached by operational research methods depends upon all relevant factors being included and measured correctly, it follows that as much attention must be paid by operational research workers to measurement in the fields of psychology and sociology, for example, as to measurement of physical quantities. Unless this is done, confidence will be reduced, and the whole purpose of operational research (which is to give confidence in decisions) defeated.

INTRODUCTION

THERE are advantages to be gained by surveying from time to time the progress operational research has made as a science; it is then possible logically to decide in which areas more emphasis should be given if advancement is to continue soundly in the future.

The choice of subject for this paper followed from taking stock in this manner. It was impossible not to be struck by the preoccupation there seems to be in operational research circles with methods (elegant or otherwise) for manipulating measurements, and, on the other hand, by the small amount of attention that appears to be given to the problems of making these measurements. There seem to be very few articles in operational research books or journals which even touch on the problem of how the factors under consideration are to be measured in practice.

This paper, therefore, expresses some personal opinions on this subject. The aim primarily has been to be provocative and to stimulate thought that may improve the situation.

BACKGROUND

The following three statements provide the basis on which the rest of the paper is built:

(a) *Operational research is objective decision making.* Two comments must be made in this context. Firstly, operational research is not necessarily merely advisory, but is a scientific method by which decisions can be taken; it is only advisory if the people who use it are advisers. Secondly, since decision making is concerned essentially with future outcomes, the measurements that are necessary to keep operational research objective must often be essentially predictive in nature;

* Based on a talk given to the Operational Research Society, London, on 2 January, 1961.

(b) *Operational research deals with systems.* These systems may be simple or complex, but in general they all contain human beings. To ignore these is to be guilty, often, of fatal sub-optimization;

(c) *Operational research is concerned with improving confidence in decisions.* A decision reached by operational research methods is not necessarily different from the decision that would be reached by other methods: a right decision is a right decision, irrespective of whether it is obtained by sticking a pin into a list of all the alternatives, or by a piece of operational research work. The basic difference is in the degree of confidence that can be placed on the correctness of the results.

It is worth while to digress for a moment on this question of confidence. In the physical sciences, facts from the real world are used as a basis on which to build hypotheses; predictions made from these hypotheses are in turn compared with other facts from the real world, and, if agreement is not complete, the hypothesis is rejected or modified. In operational research this approach is rarely possible*; an operational research model must be made and used for prediction before the "real life" situation occurs that enables a check to be made. Confidence in the model, therefore, often depends only on some intuitive assessment that it is realistic, backed up by confirmation that all relevant factors are included and measured correctly. Measurement is thus as essential in ensuring confidence as in maintaining objectivity—two of the most important requirements for any operational research study.

In this connection it is probably of value to stress the importance of following up a piece of operational research, and of checking upon the assumptions made, and model used, in the light of subsequent knowledge. Only by this means can experience be gained and used to improve the quality of further research.

PROBLEMS OF MEASUREMENT

General

It has been pointed out that one of the main requirements for confidence in an operational research study is that all relevant factors should be included and measured; also that operational research is concerned essentially with systems that include human beings. It is clear, therefore, that it is important that as much attention should be given to measuring human factors (psychological and sociological) as to measuring physical factors in the system under examination.

Admittedly some progress has been made in this field; for example, quantitative methods of selection and training are becoming widely known, and the impact of ergonomics is beginning to be felt. But generally the measurement of human and subjective quantities is in a very elementary stage compared with physical measurement; factors such as goodwill, morale, motivation, and so on, tend to be treated as "imponderables" (that is, the operational research worker ignores

* See, in particular, the article by W. J. Strauss in the *Journal of the Operational Research Society of America*..[1]

them) or, alternatively, they are in effect given some arbitrary value (a 1 in 20 chance of running out of stock is assumed reasonable as far as goodwill is concerned) and no attempt is made to include them on the balance sheet. Neither of these approaches is logical, reasonable or conducive to confidence.

There is, of course, no doubt that measurement becomes more difficult the more subjective matters are included. It would seem of value, therefore, to examine some of the problems of measurement in the physical sciences in an effort to understand what is involved, and then to determine how far the experience obtained is likely to be transferable to other fields.

Measurement in the Physical Sciences

It is pertinent, in the first place, to mention briefly the manner in which scales of measurement are usually classified.[2] Measurement can be defined as the assignment of numerals to events or objects according to rules, and the distinction between the various types of scales—nominal, ordinal, interval and ratio—is essentially on the basis of the mathematical transformations that leave the scale form invariant. In turn, these determine which statistical measures are appropriate. Table 1 summarizes the distinctions in convenient form, and provides a background for subsequent discussion.

TABLE 1. TYPES OF SCALES OF MEASUREMENT

Scale	Basic empirical operations	Allowable mathematical transformation	Example	Statistics	Remarks
Nominal	Determination of equality	Any one to one substitution	Numbering types or classes	Mode χ^2 test	
Ordinal	Determination of $>$ or $<$	$x' = f(x)$ where $f(x)$ is any monotonic increasing function	Street numbers	Median Percentiles Rank order correlation	
Interval	Determination of equality of intervals	$x' = ax+b$ $a>0$	Temperature (°C or °F) Potential energy	Mean, *s.d.*, *t*-test, *F*-test	Zero point by convention or for convenience
Ratio	Determination of equality of ratios	$x' = cx$ $c>0$	Length Temperature (°K)	Geometric mean Harmonic mean	Actual zero point

Each column is cumulative, in the sense that any statement against a given scale not only applies to that scale but to all scales below it.

The third column shows the mathematical transformations that leave the scale form invariant.

In the physical sciences measurements may also be classified according to whether they are basic (lb, ft, etc.), or derived (lb/ft^3, ft/sec). One very noticeable fact about the latter (secondary) quantities is that none is in present scientific use which does not satisfy the principle that its dimensional formula is composed of powers; that is, all are measured on ratio scales. The advantage is, of course, that the form of any mathematical expression in which these quantities are represented is independent of the precise units of measurement employed, and this flexibility has undoubtedly been responsible for the gradual elimination of measurements on scales weaker than the ratio scale.

It is also of interest to classify measurements according to the methods used to take them. These methods are either direct, as when an unknown length is measured by placing a foot rule immediately alongside, or indirect, as when a temperature, for instance, is measured in terms of the length of a column of mercury. The latter form of measurement generally implies that a quantitative relationship between the effect measured and the factor under examination is known; but this is not necessary. In fact it seems likely that in operational research indirect measurement will remain of extreme importance even if the connecting relationships are not known; the possibilities of measuring goodwill in terms of orders lost, or morale in terms of productivity are encouraging, even though the causal connections are elusive.

One particular difficulty of measurement in the physical sciences is that, in general, it is desirable to have a form of measurement that can be applied to many different systems; only then can correlations between systems be expressed in a meaningful manner. Conversely, if measurements are to be applied to only one system they can be completely arbitrary; and this is the stage that has been reached in the majority of operational research problems—normally any system of measurement used is only invented for, and intended to apply to, the problem in hand. The desirability of eventually obtaining more universality is, however, obvious.

Psycho-physical and Subjective Measurement

It is now of interest to examine, against this background, the measurement of psycho-physical magnitudes, such as brightness, loudness, apparent length, and so on, and of other subjective magnitudes such as utility.

A noticeable difference between measurement in these fields and measurement in the field of the physical sciences is the intrusion of what may be called human variability; no two people react in exactly the same way to a given stimulus, and it is therefore obviously impossible to erect a single scale of values which can satisfactorily represent all their individual feelings. The most to be hoped for is that a scale can be chosen that will correspond to the mean reaction of the population as a whole. This scale can then be used as a basis on which to compare the effects of different stimuli, or as a "norm" against which individual idiosyncrasies may be measured.

Early work was carried out by Fechner who assumed that "just noticeable differences" corresponded to equal units of sensation; he was able to relate psychological magnitude, σ, to the magnitude of the stimulus, ϕ, by the well-known equation

$$\sigma = k \log \phi \tag{1}$$

There is, however, some evidence that Fechner's assumption is not true for prothetic continua; that is, for magnitudes such as brightness or loudness in which excitement seems to be added to excitement as progress is made along the scale.[3] Just noticeable differences do not correspond to equal units of sensation in these continua, and, indeed, a relationship of the type

$$\sigma = k\phi^n \tag{2}$$

will represent results at least as well, if not better, within the limits of experimental accuracy.

The implication is that a ratio scale (as distinct from the interval scale implied by Fechner's relationship) may be used for at least some psycho-physical magnitudes; since experience in the physical sciences has indicated the power of ratio scales there seems every advantage in pursuing this lead in future.

A similar situation applies to the measurement of utility. There is an added difficulty here, of course; whereas the subjective response to, say, brightness can be tested out experimentally relatively easily, there is no method of asking for an estimate of the relative utility of £5 and £10 without mentioning the numbers 5 and 10, and this may well so bias the observer that his responses do not represent his real subjective feelings so much as a compromise between what he feels and what he thinks he ought to feel. The approach of Von Neumann and Morgenstern is one attempt to overcome this problem; but the subject of an experiment on the lines they suggest is, in effect, asked not only to make assessments of utility but also of subjective probabilities, and the practical value is thus not really clear, in spite of the elegance of the approach. Other work, using more direct methods, has suggested that a ratio scale for utility may indeed be possible[4]; again, this would seem worth pursuing.

Other Aspects of Measurement in Operational Research

There are obviously many interesting implications of the above ideas, but these cannot be pursued here. In conclusion, however, it would seem pertinent to make the following three comments which refer particularly to measurement in operational research:

(a) The difficulty of combining measurements representing different aspects of a system (such as effectiveness and cost) into a single measure of overall value is well known, and is faced daily by operational research workers. The more aspects of a system that can be measured, the more difficult this combination is likely to be; and if all measurements use different types of scale, the problem will be aggravated still further. It is important not to lose sight of this when new scales are set up.

(b) Even if scales of measurement are available for all the aspects of a system in which an operational research worker is interested, there can still be an immense practical problem in obtaining data. The value of approximate methods, inequalities, taking limits, and so on, is therefore not likely to diminish in the future.

(c) There is a tendency in operational research always to examine a system in greater and greater detail, and this is often reflected in the scales of measurement that are set up. Great value has been obtained in the physical sciences, however, from building up overall measures of the system under examination; concepts of total energy, of conservation of mass, and of statistical mechanics, for example, are extremely powerful. Perhaps more use could be made of an analogous approach in operational research.

CONCLUSIONS

To sum up: Operational research is invariably concerned with the examination of systems which include human beings. Since the degree of confidence in a decision reached by operational research depends upon all relevant factors being included and measured correctly, it follows that as much attention must be paid by operational research workers to measurement in the fields of psychology and sociology, for example, as to measurement of physical qualities. Unless this is done, confidence will be reduced and the whole purpose of operational research (which is to give confidence in decisions) defeated.

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Statistical Problems in Airline Reservation Control

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1. INTRODUCTION

AIRLINES all over the world are faced with the problem of high cancellation rates and consequent loss of revenue through airliners departing with some seats empty. When intending passengers cancel seats just before departure or even do not turn up ("no-shows"), in the knowledge that a refund of their fare will be made to them, the airline cannot always get replacements for them at such short notice. This paper describes the steps taken by one international airline to try to overcome this problem, by allowing controlled overbookings to be made and relying on subsequent cancellations to keep the number of bookings left at departure time at, or just below, the seating capacity of the plane.

The fact that a large proportion of airline seats are sold on the "Sell and Report" system, as it is called, makes it desirable that a close check is exercised on the progressive passenger loading of any particular flight. "Sell and Report" procedure, as usually employed, confirms a seat on the flight to the intending passenger before he leaves the agency premises (agents being situated all over the country or perhaps world), after which the agent records the sale of the passenger's requirements by aeradio or telegram to the Reservations Control Centre, where a seat control chart is kept for each flight. When a flight becomes fully booked, "Stop sales and waitlist" messages are sent out to agents by the control centre. The agent then requests a seat to be "wait-listed" on the passenger's first choice, and protects a seat on the next flight available for sale.

This two-stage protective action has disadvantages:

(1) the intending passenger is not able to get immediate confirmation of the flight he requires,

(2) the control centre records are embarrassed by multiple listings for the same passenger, for he can quite often be wait-listed on several other flights as well as his first choice.

These disadvantages can be obviated to a certain extent by allowing controlled overbooking on a wait-listed flight.

It is probably fair to state that most airlines do allow overbookings to be made to help cut down cancellation losses, but also that some have no exact knowledge of the risks involved. The risk, of course, is that there will not be enough cancellations so that there will still be overbookings at departure. It is desirable to keep this risk fairly small (say 0·05 or 0·01, i.e., once in 20 or 100

wait-listed flights) so as not to incur possible expense and loss of goodwill through having to offload passengers. (It is, however, shown later in the paper that even allowing quite large values for the financial penalty involved in off-loading a passenger—including loss of goodwill—a greater risk than might seem acceptable can often be run, i.e. a total financial gain can still be made with a larger number of overbookings.)

In section 2 the hypothesis is proposed that cancellations are independent, random events and the fit of some data on cancellations to this model is examined. In section 3, overbooking recommendations are made using binomial probability theory to calculate the risks involved. They cover different times before departure, all possible seating arrangements and two risks $\alpha = 0{\cdot}05, 0{\cdot}01$, so that the reservations superintendent is provided with a ready-made set of decision rules to cover most contingencies. In section 4 an analysis of the expected losses when the overbooking system is used is compared with the expected losses under the previously existing system, leading to the conclusions mentioned in the preceding paragraph. The adequacy and possible extensions of the work are also discussed.

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Most of the ground in the present paper has already been covered by Beckmann[1], but two factors seemed to make it worth while to publish this study. The greater part of it was done independently of Beckmann's work and uses a different approach, and the results, in terms of distributions obtained, are also quite different from those reported in the companion paper by Beckmann and Bobkowski[2] and used by Beckmann. They fitted gamma distributions (which are positively skewed) to demand and loss distributions, whereas the evidence in this investigation shows the demand distributions to be markedly negatively skewed, possibly because of a higher overall load factor (nearly 90 per cent in the period under examination) than the American airline. Cancellation distributions are shown to be binomial here, and although binomial and gamma distributions are fairly close sometimes, the differences are sufficiently marked for overbooking recommendations and risks to be materially affected. A major difference appears to be in the number of no-shows, which are quite rare here and have been ignored; without further evidence on the airline whose data Beckmann used it is impossible to say whether this difference is characteristic of the type and scheduling of the flights (long-haul or short-haul, frequent or infrequent, etc.) or of passenger attitudes to airline travel generally.

2. EXAMINATION OF BOOKING RECORDS

The data used in this paper comprise 59 DC6 flights from Auckland to Sydney over the months of April, May and June, 1957. There were no *a priori* reasons why the period studied should not be homogeneous with regard to bookings and cancellations, it being a period where demand for seats remained fairly steady, and there appeared to be no trends or other abnormalities so it was

treated as such. Three different seating arrangements were in use on these flights, 16 first class and 41 tourist class, 26 first and 29 tourist, and 34 first and 19 tourist. These will be referred to as 16–41, 26–29 and 34–19 respectively, and the two classes as F and T, for short. The form in which booking records were kept at the reservations control centre for this period enabled groups of passengers who were confirmed bookings at three times before departure (14, 7 and 2 days) to be followed right through to departure time and the number of cancellations from each group noted.

FIG. 1. *Booking pattern for Flight* 463, 18/5/57
B = *number of passengers booked*, C = *number of passengers cancelled*
(*See text for further explanation*)

Days, or period before departure	14	14–7		7	7–2		2	2–0		0
	B	B	C	B	B	C	B	B	C	B
14	20, 27		1, 4	19, 23		2, 0	17, 23		0, 4	17, 19
14–7		1, 6	0, 1	1, 5		0, 3	1, 2		0, 1	1, 1
7				20, 28		2, 3	18, 25		0, 5	18, 20
7–2					9, 4	2, 0	7, 4		1, 1	6, 3
2							25, 29		1, 6	24, 23
2–0								2, 4	0, 0	2, 4
0										26, 27

A typical booking pattern for a flight with seating configuration 26–29 is as shown in Figure 1, in which the first number of each pair refers to first class, the second to tourist class passengers. Each line follows the fate of passengers who were either confirmed bookings at one of the 3 check points or who booked in an intervening period. Thus, of the 47 confirmed bookings at 14 days, 5 cancelled between 14 and 7 days, 2 between 7 and 2 days, and 4 between 2 and 0, leaving 36 passengers of this group still booked at departure. The cancellation rates for this flight for the period 14–0 were therefore: F, 3/20; T, 8/27. Seven intending passengers made bookings between 14 and 7 days; 1 cancelled in the same period, 3 between 7 and 2, and 1 between 2 and 0. The first entry in the third line summarizes the position at 7 days before departure, and so on. The only other information available, apart from whether each passenger was an individual or one of a group booking, was the number of bookings cancelled more than 14 days before departure and therefore of no direct interest in the present investigation. In the example of Figure 1 there

were 15 F and 15 T in this category, which when added to the 20 subsequent cancellations gave a cancellation rate for the particular flight of 50/103. The overall cancellation level over the 59 flights of 40 per cent (F, 42 per cent; T, 38 per cent) will give the reader some idea of the cancellation problem.

The airline booking set-up has strong analogies with several processes in the physical world, such as emissions from radio-active particles and arrivals of telephone calls at an exchange. The essential feature of this type of process is that we have events occurring from time to time at more or less irregular intervals, the number occurring in any given interval following some probability distribution. In our application the events are the recording, by reservations control, of either a booking or a cancellation, and the interval we are concerned with is one of the three periods from 14, 7 or 2 days before departure to the actual day of departure. Let us then postulate that bookings and cancellations are random events, and examine the implications of the proposed model. Now it is well known that if events occur randomly and independently in time at a constant rate the number of events in intervals of the same length follows a Poisson distribution. However, in the case of cancellations we are led to consider the binomial distribution, as cancellations for a given flight can only occur from a fixed number N of confirmed bookings. The situation is then equivalent to N Bernoulli trials with a constant probability p of success, a "success" in this case being a cancellation.

One necessary assumption in the above model is that the process is stationary in the statistical sense, which means that the probability of an event is the same for all intervals of the same length, independent of the past history of the system. This can be tested by taking a group of bookings and comparing the numbers of cancellations from the group in different time intervals with expectation. Taking all passengers who were confirmed bookings at 14 days before departure as our group ($N = 2413$) we find that 230 passengers cancelled between 14 and 7 days before departure, 159 between 7 and 2, and 78 between 2 and 0, intervals of 7, 5 and 2 days, respectively. The expectations, assuming a constant rate, are 233·5, 166·8 and 66·7, respectively, and the resulting χ^2 of 2·31 with 2 degrees of freedom is not significant. Thus our process may be considered stationary.

A more likely cause of departures from the model will be the occasional lack of independence of events. This will occur in the case of block bookings and cancellations, i.e., when a party of more than one person, such as a family, is travelling. One can conceivably imagine other causes of non-independence, but they are hardly likely to rank in importance. Some, like a chartered flight, for instance, where the passengers may not be one party but are nevertheless not independent, would be automatically excluded from consideration. Others, like panic cancellations due to political conditions or a spate of air crashes, are remote possibilities. A careful examination of the series of 59 flights failed to reveal any important factor affecting independence other than block bookings, whose effect will now be assessed.

Of all the passengers who were still confirmed bookings 14 days before departure, or who booked subsequently ($N = 3816$), 62 per cent were individuals travelling alone, 23 per cent were in pairs, and only 5 per cent in parties of more than 6 passengers. Parties of 6 or less passengers actually comprised 99·6 per cent of all bookings, of which 81 per cent were individual and 15 per cent were paired bookings. The proportions for F and T considered separately were substantially the same. The cancellation rates (number of cancellations/number of bookings) were not significantly different between individuals and groups of 2–6 passengers (F only, $P = 0{\cdot}63$; T only, $P = 0{\cdot}32$; F and T combined, $P = 0{\cdot}29$). However, the party cancellation rate for parties of greater than 6 passengers was very much less, and it was thought advisable to omit these from consideration, especially since decisions on overbooking a flight would be materially altered if a large party was already booked. Large parties do not occur very frequently (there were only 11 on the 59 flights examined), but there are always smaller groups present and it would be unrealistic to omit them as well.

Table 1 summarizes the data and the tests and calculations performed on them. The goodness of fit of the cancellations to the binomial distribution has been tested by means of the χ^2 homogeneity test for proportions, which is equivalent arithmetically to testing for independence in a $2 \times n$ contingency table (here n = number of flights in a category). There are doubts concerning the adequacy of this test when there are small expectations in several of the cells and not many degrees of freedom.[4] The exact treatment of Freeman and Halton[5] is available in cases of doubt, but the amount of labour involved would be prohibitive for the data of Table 1. However, one category with half the expectations less than 1 (F, 16 seats, 2–0 days, all passengers) was subjected to the exact analysis, the resulting probability being 0·29 compared with $P(\chi^2) = 0{\cdot}28$. The example of Freeman and Halton[5] (p. 143) with only 2 degrees of freedom gave $P(\chi^2) = 0{\cdot}093$ compared with the exact probability of 0·088. These differences are negligible for our purposes, and as the example from Table 1 is one of the extreme cases, the expectations in most of the others being much greater, I think we are justified in accepting the χ^2 test as satisfactory. If a check is needed, there is a partial one in the percentage of binomial variation in the crude cancellation rates (B) which has been determined according to Cochran[3]. This is not a test, since no significance level can be attached to the result, and further with the small numbers of degrees of freedom pertaining here, B will itself be subject to large sampling fluctuations.

Turning now to a consideration of the results themselves, we find that high values of P are invariably accompanied by high values of B (all categories with $P > 0{\cdot}30$ have $B > 90$) and the reverse is also true, with one or two exceptions with high B (> 70) and low P. Cochran[3], using B to determine whether binomial or equal weights should be used in an analysis of variance of percentages based on unequal numbers, recommends as a rough guide that binomial weighting be employed if $B > 80$. If we arbitrarily take those categories with $B > 70$ as

TABLE 1. DATA AND CALCULATIONS ON BOOKINGS AND CANCELLATIONS

Class		First (F)			Tourist (T)		
Seats		16	26	34	41	29	19
No. of flights		7	35	17	7	35	17
Period (days)		(a) All passengers (parties of >6 omitted)					
14–0	N_b	107	623	321	124	727	302
	N_c	14	130	99	26	143	43
	r	0·1308	0·2087	0·3084	0·2097	0·1967	0·1424
	$P(\chi^2)$	0·73	0·022	0·64	0·030	0·047	0·10
	B	100	67	100	53	73	69
	p	0·1608	0·2109	0·2941	0·2177	0·1890	0·1576
7–0	N_b	113	754	418	157	830	325
	N_c	12	116	104	32	121	37
	r	0·1062	0·1538	0·2488	0·2038	0·1458	0·1138
	$P(\chi^2)$	0·74	0·57	0·10	0·017	0·12	0·083
	B	100	100	75	41	80	85
	p	0·1078	0·1579	0·2411	0·1778	0·1492	0·1177
2–0	N_b	109	803	445	166	860	329
	N_c	4	49	54	13	56	18
	r	0·0367	0·0610	0·1213	0·0783	0·0651	0·0547
	$P(\chi^2)$	0·29	0·050	0·00001	0·91	0·0023	0·14
	B	82	69	38	100	45	72
	p	0·0055	0·0556	0·1388	0·0970	0·0683	0·0369
		(b) Single passengers only					
14–0	N_b	58	384	210	65	422	200
	N_c	10	81	69	10	91	31
	r	0·1724	0·2109	0·3286	0·1538	0·2156	1550
	$P(\chi^2)$	0·74	0·19	0·28	0·32	0·25	0·27
	B	100	85	89	91	86	87
	p	0·2045	0·2110	0·3195	0·1819	0·2061	0·1659
7–0	N_b	64	482	279	79	505	220
	N_c	10	71	75	11	91	28
	r	0·1563	0·1473	0·2688	0·1392	0·1802	0·1273
	$P(\chi^2)$	0·50	0·72	0·075	0·24	0·50	0·062
	B	100	100	74	79	100	64
	p	0·1463	0·1529	0·2614	0·1509	0·1751	0·1349
2–0	N_b	62	531	294	90	529	227
	N_c	4	30	43	9	43	16
	r	0·0645	0·0565	0·1463	0·1000	0·0813	0·0705
	$P(\chi^2)$	0·43	0·020	0·0026	0·65	0·40	0·26
	B	100	58	52	100	97	85
	p	0·0447	0·0513	0·1598	0·0695	0·1349	0·0535

N_b = Number of passengers booked.
N_c = Number of passengers cancelled.
r = Crude cancellation rate (N_c/N_b).
$P(\chi^2)$ = Probability of χ^2 homogeneity test.
B = Percentage of binomial variation.
p = Smoothed cancellation rate.

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satisfactory, we find 11 out of 18 in the top half of Table 1 (all passengers) and 15 out of 18 in the lower half (single passengers only), which compares extremely well with the 11 and 16 respectively satisfactory on the basis of $P(\chi^2)$, using 0·05 as the discriminating point. The unfavourable results would indicate that the hypothesis of binomial distribution of cancellations is conclusively rejected when parties are included, although possibly acceptable for single passengers only. However, a closer look at the figures indicates that this is not completely true.

The χ^2 test is very critically affected by the occasional extreme deviation. There are only 3 low values of $P(\chi^2)$, in the lower half of the table, 0·0026, 0·020 and 0·062. In each case these are due to a single flight with anomalous results, which are respectively 4 cancellations from 6 bookings, 6/20 and 7/17. With these flights omitted and χ^2 recalculated, $P(\chi^2) = 0{\cdot}084$, 0·34 and 0·49, respectively, indicating that apart from the three flights, the cancellation rates for individuals are indeed constant within each category. For all passengers the same is substantially true, although there is no doubt as to the overall effect of groups. Seven out of the 18 categories have $P(\chi^2) \leqslant 0{\cdot}05$, and in 6 of these 7 the omission of one apparently anomalous flight (two in one case), where party cancellations made the number of cancellations too high, removed the significance. For example, the category (F, 26, 14–0) has one flight with 7 cancellations from 11 bookings. These include 2 parties of 2 persons, and with this flight omitted, $P(\chi^2) = 0{\cdot}15$. Again in (T, 29, 14–0) there is one flight with 14/28, including parties of 4 and 6 persons, and the new $P(\chi^2) = 0{\cdot}42$. (Note, however, from the bottom half of Table 1 that with parties removed, these particular flights are not now anomalous.) We conclude that this analysis, if interpreted liberally, shows that the party bookings do not disturb the randomness sufficiently for it to matter, and that to a first approximation anyway we may regard cancellations as distributed binomially, even when parties (of $\leqslant 6$ passengers) are present.*

The smoothed cancellation rates (p) were calculated by a least squares weighted analysis on each block of 9 categories, fitting constants for seating arrangements and periods. In every case period differences were very highly significant, as would be expected since there are more cancellations in a longer period of time, and in no case was the interaction between periods and seating arrangements greater than the error mean square. The surprising feature of the analysis is the significance of seating arrangements, which is very high ($P < 0{\cdot}001$) for F and marginal ($P = 0{\cdot}025$) for T. One would intuitively expect the cancellation rates to be the same for all seating arrangements within each period and class, but it appears that a higher cancellation rate is to be expected when there are more seats available, the differences being strongly marked for F and smaller, though still present, for T. Now the seating arrangement on a

* It is conceivable that the variation between individual passengers is largely binomial, with a small admixture of some extraneous factor which, while small as a contribution to the whole distribution, might make an important contribution to the χ^2 test.

flight is determined by the demand, and may even be changed during the course of booking, say from 26–29 to 34–19 if in this case more F seats than expected are being booked. The only explanation that can be offered for this phenomenon is that a smaller number of seats available inhibits the usual tendency to cancel, or alternatively that a larger number of seats gives this tendency free rein!

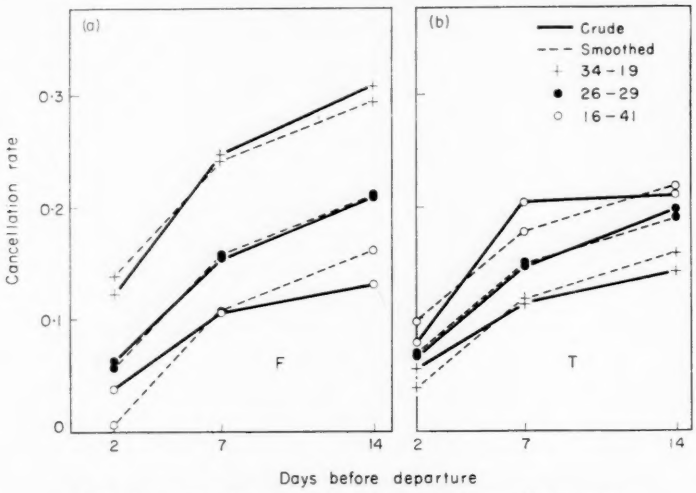

FIG. 2. *Comparison of crude and smoothed cancellation rates.*

It would presumably be quite satisfactory to use the crude cancellation rates for the subsequent work on overbooking recommendations and losses, but the smoothed rates do, as their name implies, remove some of the inconsistencies which arise from the small numbers in some of the categories, especially 16–41 with only 7 flights. Furthermore, the overall closeness of the smoothed and crude rates in all categories means that recommendations would be substantially the same whichever set were used. It is possible that some mathematical function could be chosen which fits the crude rates more closely—an examination of Figure 2 suggests, for example, a family of exponential curves—but the complete absence of interaction between periods and seating arrangements does not nullify the linear hypothesis. Finally, it should be noted that the rates for all passengers are also close to those for individuals only, the differences overall being not significant. This, when considered in conjunction with the results from the χ^2 tests, provides sufficient justification, I feel, for the acceptance of the model proposed.

3. OVERBOOKING RECOMMENDATIONS AND RISKS

Given a flight with a capacity of N_1 F seats and N_2 T seats, the time before departure and the probabilities p_1 and p_2 of F and T cancellations respectively for this period and seating arrangement, we wish to determine the risk attached

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to allowing n_1 extra F bookings and n_2 extra T bookings or, alternatively, how many extra F and T bookings can be allowed so that the risk equals some pre-assigned value α. The risk is here defined as the probability that a flight will still be overbooked at departure. We are not concerned at the moment with how many overbookings remain, as the next section deals with this point. The risk is therefore simply the sum of the (independent) probabilities of all the unfavourable cases, which are (i) too few F and too few T cancellations, (ii) too few T but sufficient F cancellations, (iii) too few F but sufficient T cancellations.

Let us assume first that an overbooked F passenger is not willing to be transferred to an empty T seat (we relax this restriction slightly later) but that an overbooked T passenger will not object to being transferred to an empty F seat in an emergency. This seems reasonable, since F passengers are willing initially to pay the difference between F and T fares to obtain a better seat and therefore should not be so willing to forgo this advantage. In fact, however, reservations control staff with whom I have discussed this problem, have assessed the chances as about even that an overbooked F passenger will accept an empty T seat (and of course the consequent fare adjustment also). For the sake of generality in treatment, we will allow a probability k $(0 \leqslant k \leqslant 1)$ that an overbooked F passenger will accept an empty T seat (later considering only $k = 0, \frac{1}{2}$), and denote by R_k the risk, for given k, that there are still overbookings at departure.

Allow n_1 F and n_2 T overbookings, and let i,j represent the number of F and T cancellations respectively ($i = 0, 1, \ldots, N_1+n_1$; $j = 0, 1, \ldots, N_2+n_2$). The unfavourable cases to the airline, assuming $k = 0$, are:

(1) $i<n_1, j<n_2$—there remain n_1-i F and n_2-j T overbookings at departure;

(2) $n_1 \leqslant i<n_1+n_2, i+j<n_1+n_2$—there remain n_1+n_2-i-j T overbookings, $i-n_1$ having been transferred to F seats;

(3) $i<n_1, j \geqslant n_2$—there remain n_1-i F overbookings, since no transfer from F to T is allowed.

The risk of overbookings at departure is found by adding these probabilities, since they are all independent, and is

$$R_0 = \sum_{0}^{n_1-1} \sum_{0}^{N_2+n_2} p_{ij} + \sum_{n_1}^{n_1+n_2-1} \sum_{0}^{n_1+n_2-i-1} p_{ij}$$

$$= \sum_{0}^{n_1-1} p_{i\cdot} + \sum_{n_1}^{n_1+n_2-1} \sum_{0}^{n_1+n_2-i-1} p_{ij} \qquad \text{(a)}$$

where

$$p_{ij} = \{b(i;\ N_1+n_1, p_1)\}\{b(j;\ N_2+n_2, p_2)\}$$

$$p_{i\cdot} = b(i;\ N_1+n_1, p_1)$$

$$b(r;\ n, p) = \binom{n}{r} p^r (1-p)^{n-r}$$

If we allow $k = 1$, i.e. all overbooked F passengers willing to transfer to empty T seats, the unfavourable cases are simply when $i+j<n_1+n_2$, and the risk is

$$R_1 = \sum_{0}^{n_1+n_2-1} \sum_{0}^{n_1+n_2-i-1} p_{ij}$$

The difference from R_0 is that part of (a) for which $i+j \geqslant n_1+n_2$.

For the general case, the chance that not all of n_1-i overbooked F passengers will take a T seat is $1-k^{n_1-i}$. The risk generally is therefore

$$R_k = R_1 + \sum_{i=0}^{n_1-1} (1-k^{n_1-i}) \sum_{j=n_1+n_2-i}^{N_2+n_2} p_{ij}$$

The answers to both questions posed at the beginning of this section must be obtained in the same way, since n_1 and n_2 cannot be solved explicitly in terms of α and the other constants. n_1 and n_2 are found by trial-and-error methods, substituting different sets (n_1, n_2) in the equations and calculating R_k. We then use as our recommendation the set whose risk is nearest to α and preferably below it. Alternatively, if R_k is too different from α, as a refinement we can take the set whose risk is nearest below α ($\alpha - R_k = d_1$ say) and the set whose risk is nearest above α ($R'_k - \alpha = d_2$ say) and use these sets at random in the proportion $d_2 : d_1$. The risk will then be on the average exactly α.

The recommendations for the two cases $k = 0, \frac{1}{2}$ and for $\alpha = 0{\cdot}05, 0{\cdot}01$ are given in Table 2. They have been chosen as far as possible so that the marginal probabilities of F and T overbookings $\left(\sum_{0}^{n_1-1} p_{1\cdot} \text{ and } \sum_{0}^{n_2-1} p_{\cdot j} \text{ respectively} \right)$ are approximately equal. The same risks could be achieved by increasing n_2 and decreasing n_1; for example, in the category 26–29, 14 days, the recommendation 1–5 as well as 2–2 gives $R_k = 0{\cdot}0118$. However, this is unbalanced, in the sense that we would be relying to a much greater extent on the F cancellations to absorb the greater expected numbers of T overbookings, and it is desirable to keep the number of interchanges between F and T to a minimum. Sometimes it has been necessary to allow no overbookings in one class (or both). This occurs when the probability of no cancellations in the particular category (or p_{00}) is too high. In the category 26–29, 2 days, the risk of 0–2 (0·1211) is the nearest above the 0·05 level. It may, however, be considered too high to use approximately one-quarter of the time ($= (0{\cdot}05-0{\cdot}0271) \div (0{\cdot}1211-0{\cdot}0271)$ where 0·0271 is the risk for 0–1); in this case 0–1 would have to be used all the time, with a lower risk than the nominal 0·05.

Some features of Table 2 are worth commenting upon. It would appear that 2 days before departure is rather too late to be useful, except possibly for 34–19. At the 0·01 level, 40 per cent of the time no overbookings at all can be allowed, if account is taken of the relative proportions of occurrence of the 3 seating arrangements in the sample (from Table 1). A further 51 per cent of the time

only 1 overbooking is permissible. Even at the 0·05 level 1 overbooking only is permissible 48 per cent of the time (63 per cent if we exclude the high risk 0·1211 of 0–2). The surprising feature to most readers will probably be the large number of overbookings that can be allowed at fairly small risks, due of course to the high cancellation rates and consequent large expected number of cancellations in most categories.

TABLE 2. OVERBOOKING RECOMMENDATIONS AND RISKS FOR $k = 0, \frac{1}{2}$

Seating arrangement		16–41	26–29	34–19
Days before departure	k	$n_1 - n_2$ (R_k)	$n_1 - n_2$ (R_k)	$n_1 - n_2$ (R_k)
14	0	0–6 (·0066) 0–7 (·0154) 1–3 (·0511)	2–2 (·0118) 3–3 (·0439) 3–4 (·0533)	6–0 (·0106) 8–1 (·0464) 8–2 (·0533)
	$\frac{1}{2}$	0–6 (·0066) 0–7 (·0154) 1–6 (·0392) 1–7 (·0544)	2–3 (·0092) 3–4 (·0403) 4–3 (·0702)	6–2 (·0112) 8–2 (·0432) 8–3 (·0635)
7	0	0–4 (·0079) 0–5 (·0217) 0–6 (·0487)	1–1 (·0101) 2–1 (·0517)	4–0 (·0099) 5–2 (·0418) 6–1 (·0611)
	$\frac{1}{2}$	As for $k = 0$	1–2 (·0084) 1–3 (·0178) 2–2 (·0369) 2–3 (·0556)	4–1 (·0088) 6–1 (·0473)
2	0	0–0 (—) 0–1 (·0126) 0–2 (·0650)	0–0 (—) 0–1 (·0271) 0–2 (·1211)	1–0 (·0053) 1–1 (·0196) 2–0 (·0313) 2–1 (·0669)
	$\frac{1}{2}$	As for $k = 0$	As for $k = 0$	1–0 (·0040) 1–1 (·0191) 2–0 (·0243) 2–1 (·0646)

4. LOSSES THROUGH OVERBOOKING

For a complete discussion of the overbooking problem some mention must be made of the possible financial losses which may be incurred through using the recommendations of the last section. This is rather a difficult question, as it is virtually impossible in practice to place an exact value on the financial penalty incurred when an overbooked passenger has to be offloaded. It may involve hotel and other incidental expenses while the passenger awaits another flight, but the greatest contribution will almost certainly be loss of goodwill. Just how

many potential travellers will be affected by adverse publicity given to the airline by a disgruntled offloaded passenger must remain a matter for conjecture, although the number will possibly be higher if the passenger knew he had been overbooked deliberately rather than through a clerical error. Airlines, judged by their reluctance to entertain more than a bare minimum (unstated) of these cases, must place a very high (but again unstated) value on the penalty. In what follows we allow a quite arbitrary value c for the penalty per offloaded passenger, and determine a limit for c below which the total expected loss due to overbooking is less than the expected loss due to unsold seats only if no overbookings were made.

Let the first and tourist class fares be f_1 and f_2. These will be taken as the losses on an empty F or T seat respectively. Then with the same notation as the previous section, we have the following unfavourable cases:

(1) $i<n_1, j\leqslant n_2$. There remain n_1-i F and n_2-j T overbookings at departure, and the penalty is $c(n_1+n_2-i-j)$;

(2) $n_1\leqslant i<n_1+n_2, i+j<n_1+n_2$. There remain n_1+n_2-i-j T overbookings since $i-n_1$ are transferred to empty F seats at a loss of fare of f_1-f_2 per seat. The loss is therefore $c(n_1+n_2-i-j)+(f_1-f_2)(i-n_1)$;

(3) $i<n_1, j>n_2$. There are n_1-i F overbookings and $j-n_2$ empty T seats. If no transfer from F to T is allowed, the loss is $c(n_1-i)+f_2(j-n_2)$.

If overbooked F passengers have a constant chance k of accepting empty T seats, there are two possibilities:

(i) $i<n_1, i+j\geqslant n_1+n_2$. The probability of r of the n_1-i F overbookings accepting empty T seats is

$$\pi_r=\binom{n_1-1}{r}k^r(1-k)^{n_1-i-r}$$

$$=\binom{n_1-1}{r}(\tfrac{1}{2})^{n_1-i} \qquad \text{if } k=\tfrac{1}{2}.$$

The expected loss, averaging over all r, is

$$c(n_1-i)(1-k)+f_2\{j-n_2-k(n_1-i)\}.$$

(ii) $i<n_1, i+j<n_1+n_2$. Not all of the n_1-i F overbookings can accept empty T seats even if they are willing to. If r of the n_1-i accept, the loss is $c(n_1-i-r)+f_2(j-n_2-r)$ with probability $\pi_r, r=0,1,\ldots,j-n_2-1$,

$$1-\sum_0^{j-n_2-1}\pi_i, r=j-n_2.$$

Averaging over all r, the expected loss is

$$c(n_1-i-\bar{r}_{ij})+f_2(j-n_2-\bar{r}_{ij})$$

where

$$\bar{r}_{ij}=\sum_0^{j-n_2-1}r\pi_r+(j-n_2)\left(1-\sum_0^{j-n_2-1}\pi_r\right).$$

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A short table of $\bar{r}_{ij}$ for $k = \frac{1}{2}$ follows:

n_1-i \ $j-n_2$	1	2	3	4	5	6	Divisor
7	127	247	346	410	439	447	128
6	63	120	162	184	191		64
5	31	57	73	79			32
4	15	26	31				16
3	7	11					8
2	3						4

(4) $i \geqslant n_1, i+j \geqslant n_1+n_2$. There are $i-n_1$ empty F seats, $j-n_2$ empty T seats and the loss is $f_1(i-n_1)+f_2(j-n_2)$. [If $j<n_2$, the n_2-j overbooked T passengers are transferred to F seats at a loss of f_1-f_2 per seat, and there remain $i-n_1-(n_2-j)$ empty F seats, so the loss is

$$f_1(i+j-n_1-n_2)+(f_1-f_2)(n_2-j) = f_1(i-n_1)+f_2(j-n_2).]$$

Multiplying each value by the appropriate binomial probability p_{ij} of i F and j T cancellations and collecting terms, we obtain the total expected loss through overbooking for the general case as

$$\begin{aligned} L_k = c\Bigg[& \sum_{0}^{n_1-1}\sum_{0}^{n_2}(n_1+n_2-i-j)p_{ij} + \sum_{n_1}^{n_1+n_2-1}\sum_{0}^{n_1+n_2-i-1}(n_1+n_2-i-j)p_{ij} \\ & + \sum_{0}^{n_1-1}\sum_{n_2+1}^{N_2+n_2}(n_1-i)p_{ij} - \sum_{0}^{n_1-1}\sum_{n_2+1}^{n_1+n_2-i-1}\bar{r}_{ij}p_{ij} - k\sum_{0}^{n_1-1}\sum_{n_1+n_2-i}^{N_2+n_2}(n_1-i)p_{ij}\Bigg] \\ & + f_1\Bigg[\sum_{n_1}^{n_1+n_2-1}\sum_{0}^{n_1+n_2-i-1}(i-n_1)p_{ij} + \sum_{n_1}^{N_1+n_1}\sum_{n_1+n_2-i}^{N_2+n_2}(i-n_1)p_{ij}\Bigg] \\ & + f_2\Bigg[\sum_{0}^{n_1-1}\sum_{n_2+1}^{N_2+n_2}(j-n_2)p_{ij} + \sum_{n_1}^{N_1+n_1}\sum_{n_1+n_2-i}^{N_2+n_2}(j-n_2)p_{ij} \\ & - \sum_{n_1}^{n_1+n_2-1}\sum_{0}^{n_1+n_2-i-1}(i-n_1)p_{ij} - \sum_{0}^{n_1-1}\sum_{n_2+1}^{n_1+n_2-i-1}\bar{r}_{ij}p_{ij} \\ & - k\sum_{0}^{n_1-1}\sum_{n_1+n_2-i}^{N_2+n_2}(n_1-i)p_{ij}\Bigg] \end{aligned}$$

which reduces, after manipulation, to

$$L_k = c(S_1+S_2-S_3)+f_1(m_1-n_1+S_1)+f_2(m_2-n_2+S_2-S_3)$$

where $S_1 = \sum_{0}^{n_1-1} (n_1-i)p_{i\cdot}$

$$S_2 = \sum_{0}^{n_1-1}\sum_{0}^{n_2-1}(n_2-j)p_{ij} + \sum_{n_1}^{n_1+n_2-1}\sum_{0}^{n_1+n_2-i-1}(n_1+n_2-i-j)p_{ij}$$

$$S_3 = k\sum_{0}^{n_1-1}\sum_{n_1+n_2-i}^{N_2+n_2}(n_1-i)p_{ij} + \sum_{0}^{n_1-1}\sum_{n_2+1}^{n_1+n_2-i-1}\bar{r}_{ij}p_{ij}$$

$$m_1 = \sum_{0}^{N_1+n_1} ip_{i\cdot} = (N_1+n_1)p_1$$

$$m_2 = \sum_{0}^{N_2+n_2} jp_{\cdot j} = (N_2+n_2)p_2$$

We note in passing that the coefficient of c, $S_1+S_2-S_3$, is the expected number of overbookings remaining at departure; denote this number by $E_k(n)$.

For purposes of comparison with the scheme of not overbooking, i.e. normal operation, conditions of less than 100 per cent demand are not considered, because they do not qualify for overbooking. The expected loss through not overbooking is simply

$$\begin{aligned} L &= \sum_{0}^{N_1}\sum_{0}^{N_2}(if_1+jf_2)p'_{ij} \\ &= f_1\sum_{0}^{N_1} ip'_{i\cdot} + f_2\sum_{0}^{N_2} jp'_{\cdot j} \\ &= f_1m'_1 + f_2m'_2, \end{aligned}$$

where $p'_{i\cdot}, p'_{\cdot j}, p'_{ij}, m'_1, m'_2$ are the binomial probabilities and means associated with N_1 and N_2 seats only and the same probabilities of cancellation as above. The problem can then be put in either of the two following forms:

(i) given n_1 and n_2, for what values of c (in terms of f_1 and f_2) is $L_k < L$?

(ii) given c, what values of n_1 and n_2 make $L_k < L$?

In the airline whose data have been used, $f_1 = 1{\cdot}2f_2$, so for convenience the penalty is subsequently given in multiples of the tourist fare f_2. As in the previous section, the solution is a trial-and-error one. Various sets (n_1, n_2) are taken and the value of c for which $L_k = L$ is determined, the calculations being fairly easily systematized. Then for all values of c less than this calculated value, $L_k < L$ and overbooking is the more profitable system to use. In Table 3 are given the marginal values of the penalty (i.e. values of c/f_2 for which L_k exactly equals L) for the recommendations of Table 2.

We immediately notice that for most of the recommendations the permissible penalty before overbooking becomes unprofitable is very high indeed. As I stated earlier, just what is a reasonable value of the penalty is impossible to say,

but certainly values greater than 100 times the tourist class fare must surely be considered unreasonable. These would imply, for a fare of say £25, that the financial penalty for each offloaded passenger is in excess of £2500. At 14 or 7 days before departure, therefore, we can reasonably say that a risk of one in

TABLE 3. VALUES OF THE PENALTY c (IN MULTIPLES OF f_2) FOR WHICH $L_k = L$, AND $E_k(n)$, FOR THE OVERBOOKING RECOMMENDATIONS OF TABLE 2

Seating arrangement		16–41			26–29			34–19		
Days before departure	k	n_1-n_2	c/f_2	$E_k(n)$	n_1-n_2	c/f_2	$E_k(n)$	n_1-n_2	c/f_2	$E_k(n)$
14	0	0–6	512	0·01	2–2	263	0·01	6–0	344	0·01
		0–7	236	0·02	3–3	91	0·06	8–1	99	0·08
		1–3	64	0·05	3–4	81	0·07	8–2	93	0·09
	½	0–6	512	0·01	2–3	401	0·01	6–2	430	0·02
		0–7	236	0·02	3–4	108	0·06	8–2	123	0·07
		1–6	110	0·05	4–3	66	0·09	8–3	82	0·11
		1–7	85	0·08						
7	0	0–4	322	0·01	1–1	183	0·01	4–0	287	0·01
		0–5	134	0·03	2–1	46	0·06	5–2	80	0·08
		0–6	65	0·08				6–1	65	0·10
	½	As for $k = 0$			1–2	293	0·01	4–1	412	0·01
					1–3	159	0·02	6–1	92	0·07
					2–2	85	0·04			
					2–3	61	0·07			
2	0	0–0	—	—	0–0	—	—	1–0	192	0·01
		0–1	71	0·01	0–1	33	0·03	1–1	89	0·02
		0–2	23	0·08	0–2	13	0·15	2–0	56	0·04
								2–1	34	0·09
	½	As for $k = 0$			As for $k = 0$			1–0	258	0·01
								1–1	91	0·02
								2–0	75	0·03
								2–1	35	0·08

100 is too low and that the 0·05 recommendations, where the marginal penalty is in the range 50–100 times the tourist fare, should be used instead. At 2 days, the marginal penalties for the 0·05 recommendations are less than $50f_2$, so that at this level of penalty only 0·01 recommendations could be used, which restricts the utility of 2-day overbooking to the 34–19 category.

If, however, a penalty of $50f_2$ is also considered unreasonably high, then obviously it should be quite safe to allow more overbookings and accept the consequent higher risks. For this reason Table 4 has been constructed, in which

TABLE 4. VALUES OF n_1, n_2 (WITH R_k AND $E_k(n)$ IN BRACKETS) WHICH FOR GIVEN VALUES OF THE PENALTY c (IN MULTIPLES OF f_2) MAKE $L_k < L$

Penalty (c/f_2)			200	100	50	20	10	5
k	Seating	Days						
0	16–41	14	0–7 (0·0154,0·02)	0–8 (0·0317,0·05)	1–7 (0·0760,0·10)	2–8 (0·2447,0·38)	3–9 (0·4718,0·92)	4–11 (0·7197,2·03)
		7	0–4 (0·0079,0·01)	0–5 (0·0217,0·03)	0–6 (0·0487,0·08)	1–6 (0·2058,0·28)	1–8 (0·3258,0·61)	2–9 (0·6159,1·47)
		2	—	—	0–1 (0·0126,0·01)	0–2 (0·0650,0·08)	0–3 (0·1764,0·25)	0–4 (0·3380,0·57)
	26–29	14	2–4 (0·0194,0·03)	2–5 (0·0319,0·05)	3–5 (0·0725,0·12)	5–4 (0·2110,0·37)	6–6 (0·4059,0·94)	7–8 (0·6453,2·08)
		7	1–2 (0·0126,0·01)	1–3 (0·0213,0·03)	2–2 (0·0567,0·07)	3–3 (0·1711,0·25)	4–4 (0·3571,0·68)	5–5 (0·5759,1·42)
		2	—	—	—	0–1 (0·0271,0·03)	0–2 (0·1211,0·15)	1–1 (0·2543,0·28)
	34–19	14	6–3 (0·0217,0·03)	8–1 (0·0464,0·08)	9–2 (0·0900,0·16)	11–3 (0·2355,0·54)	13–3 (0·3981,1·09)	15–5 (0·6786,2·66)
		7	4–2 (0·0189,0·03)	5–0 (0·0259,0·04)	6–2 (0·0804,0·13)	8–1 (0·1816,0·35)	9–3 (0·3695,0·91)	10–4 (0·5556,1·72)
		2	—	1–0 (0·0053,0·01)	2–0 (0·0313,0·04)	3–0 (0·0962,0·13)	4–0 (0·2074,0·32)	5–1 (0·4409,0·90)
$\frac{1}{2}$	16–41	14	0–7 (0·0154,0·02)	0–8 (0·0317,0·05)	1–8 (0·0800,0·13)	2–9 (0·2274,0·41)	3–10 (0·4347,0·96)	4–11 (0·6429,1·79)
		7	0–4 (0·0079,0·01)	0–5 (0·0217,0·03)	0–6 (0·0487,0·08)	1–6 (0·1517,0·22)	2–7 (0·3952,0·71)	2–9 (0·5520,1·37)
		2	—	—	0–1 (0·0126,0·01)	0–2 (0·0650,0·08)	0–3 (0·1764,0·25)	0–4 (0·3380,0·57)
	26–29	14	2–4 (0·0154,0·02)	3–4 (0·0403,0·06)	4–4 (0·0875,0·13)	5–5 (0·1992,0·36)	6–6 (0·3614,0·81)	7–8 (0·6239,2·00)
		7	1–2 (0·0084,0·01)	1–3 (0·0178,0·02)	2–3 (0·0556,0·07)	3–4 (0·1821,0·30)	4–4 (0·3049,0·57)	5–5 (0·5285,1·24)
		2	—	—	—	0–1 (0·0271,0·03)	0–2 (0·1211,0·15)	1–2 (0·3284,0·47)
	34–19	14	7–2 (0·0242,0·04)	8–2 (0·0432,0·07)	9–3 (0·1014,0·19)	11–3 (0·2134,0·47)	13–4 (0·4334,1·27)	15–5 (0·6658,2·61)
		7	5–0 (0·0167,0·02)	6–0 (0·0375,0·05)	7–1 (0·0882,0·14)	8–2 (0·1914,0·37)	9–3 (0·3537,0·86)	10–4 (0·5465,1·68)
		2	1–0 (0·0040,0·01)	—	2–0 (0·0243,0·03)	3–0 (0·0778,0·10)	4–1 (0·2770,0·48)	5–1 (0·4262,0·86)

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c/f_2 is the independent variable and the second question above is answered. In this table each recommendation is such that its marginal value of c/f_2 is as near as possible to the boundary value at the head of the table and on the upper side of it. Thus for a given boundary value of c/f_2, all recommendations to the left of this value will have $L_k < L$.

The table shows how high are the risks that can be taken; a superficial reading would indicate that even if c is as high as $10f_2$ the airline can afford to offload passengers more than once in every three overbooked flights (considering only 14- and 7-day recommendations) and still be better off financially than by not overbooking. However, this argument overlooks the fact that $E_k(n)$ is also high, of the order of 1 for 14-day and 0·7 for 7-day recommendations, and no airline would tolerate this number of offloaded passengers every overbooked flight. An airline, if operating efficiently, would presumably want to be in the position of requiring controlled overbookings on most flights, but not necessarily all (if the need for it were either too great or too small, the scheduling of flights would be at fault). At the rate of about one per overbooked flight, therefore, the numbers of offloaded passengers and consequent complaints would mount alarmingly. In these circumstances, the value of c could even be altered, more offloadings producing a higher value of c, thereby automatically excluding the lower values of c/f_2 in Table 4. If indeed, this were the case, that the penalty depended on the cumulative number of offloadings over a period of time, then it would be a cogent reason for limiting the risk to the usual 0·05 or 0·01 levels where $E_k(n)$ is of the same order as R_k and offloadings would be reasonably rare events.

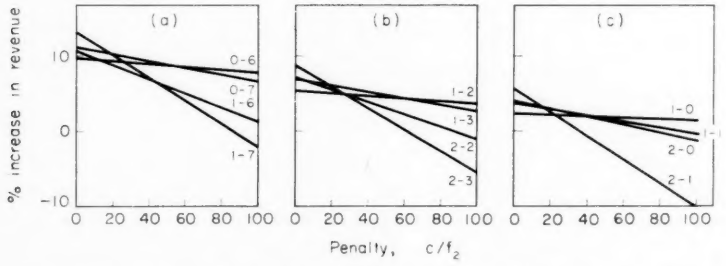

FIG. 3. *Gain in revenue as a function of penalty per overbooked passenger for* (*a*) 16–41, 14 *days*; (*b*) 26–29, 7 *days*; (*c*) 34–19, 2 *days*.

It is also interesting to see how much effect the allowing of overbookings has on the total revenue. Under the idealized scheme being discussed, in which replacements of cancellations are not considered, the revenue or gain from fares if no overbookings are allowed is:

$$G = N_1 f_1 + N_2 f_2 - L$$
$$= (N_1 - m_1')f_1 + (N_2 - m_2')f_2,$$

whereas allowing overbookings it is

$$G_k = N_1 f_1 + N_2 f_2 - L_k$$
$$= (N_1 + n_1 - m_1 - S_1) f_1 + (N_2 + n_2 - m_2 - S_2 + S_3) f_2 - E_k(n)\, c.$$

The gain (or loss) in revenue through controlled overbooking, $G_k - G$, may be expressed as a percentage of G, and in Figure 3 some typical cases have been plotted as a function of the penalty. These merely present some of the figures in Table 3 in another way, but they do show more clearly the nett effect to the airline.

5. DISCUSSION AND CONCLUSIONS

It should be emphasized first that the figures in the previous section present the case for overbooking in a rather more favourable light than would be so in practice. For, in the foregoing, "normal" operation has in fact been taken to mean closing the flight at 14, 7 or 2 days before departure if it is fully booked then, and not replacing subsequent cancellations from a wait-list, as would actually be done. The effect of this simplification has been to increase the expected loss through not overbooking, L; under actual operating conditions late cancellations and "no-shows" (intending passengers who for a variety of reasons, including missed connections, do not turn up at the airport) would tend to be the main cause of loss, since earlier cancellations would be mostly replaced from the wait-list. The chance of a wait-listed passenger accepting a seat which becomes vacant will decrease as the time before departure decreases, so that it is more advantageous to fill these seats as soon as possible, which is automatically done using controlled overbooking. Thus the contribution to the loss through overbooking, L_k, due to empty seats will still tend on the average to be less than that under true normal operation. The probable effect will be to lower the marginal value of the penalty c, but since for the 0·05 and 0·01 overbooking recommendations this marginal value is so high (except possibly for 2-day recommendations), it is safe to say that for reasonable values of the penalty the use of the recommendations in Table 2 will lead to an overall gain in revenue, although the percentage gain will be less. It is important to note, however, that the overbooking risks themselves will be unaffected by this discussion.

A truer comparison between the two systems of booking would involve the following considerations. We would first require, in comparing a given recommendation (n_1, n_2) with normal operation, that at the time before departure to which this recommendation applies the flight is fully booked and there are at least n_1 F and n_2 T intending passengers on the wait-list. (Otherwise we would not be able to overbook to this extent.) If the number of cancellations from the N_1 confirmed F bookings, r say, is not greater than n_1, their replacement is automatic, provided those on the wait-list do not either cancel or refuse a seat

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through having made other arrangements. If $r > n_1$, the number of replacements will depend on the distribution of additional demand. Thus information would be required on this distribution, i.e. on the change in the length of the wait-list with time before departure, and the cancellation rate of those wait-listed passengers who are added to the flight. Non-acceptance of a vacant seat would also count as a cancellation. It would also be necessary to investigate whether this cancellation rate changed as departure approached. It is hoped, with the co-operation of the airline, to carry out in the future such an investigation, and obtain figures not available at present.

The treatment presented in this paper is still only a partial solution of the overbooking problem, and in addition to the investigation proposed above many other topics could well be looked into. Whether or not they would be profitable would be a question for the airline itself to decide. The most important is most probably the question of different overbooking recommendations for different periods of the year; do cancellation rates change, periodically, enough to change appreciably the risks attached to the present recommendations? Another is the provision of recommendations for each class separately, to cover cases when one class only in a flight is available for overbooking. Such cases might occur when there is insufficient demand, or a large party booking, in the other class. The recommendations derived from Section 3 only apply when both classes are fully booked, and could not be applied indiscriminately to either class alone. It might also be desirable to have recommendations incorporating party booking, i.e. when one class can be partially overbooked. These ramifications, however, would all tend to make the overbooking rules less flexible, and for efficient use by an airline it is desirable to keep the rules as simple as possible.

ACKNOWLEDGEMENTS

I wish to thank Tasman Empire Airways Ltd. for permission to use the data on bookings, and more particularly Mr. C. J. W. Barton of TEAL for bringing the problem to my attention and for several useful discussions. I am also grateful to Miss M. R. K. Ward for assistance with the computations.

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Letters to the Editor

Optimal Fleet Size

INVESTIGATIONS into the optimal size of a fleet of barges owned by a Company when further barges are available for hire have led to conclusions which may have widespread application. The situation is somewhat similar to that considered by Kirby[1].

Suppose the demand for transport capacity is a periodic function whose fluctuation over one cycle is shown in the diagram.

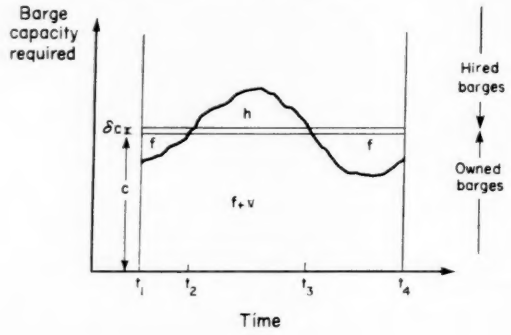

FIG. 1.

The following notation will be used:

c = total capacity of barges owned by the Company

f = fixed cost per unit capacity per day for Company-owned barges

v = variable cost per unit capacity per day for Company-owned barges (applicable only when the barge is in use)

h = cost per unit capacity per day for hired barges.

The total transport cost over the period is obtained by multiplying each area by the cost associated with it as shown in the diagram and summing.

If the capacity of the fleet is taken to be a continuous variable it can be augmented by a small amount δc causing the transport cost to rise by

$$f(t_2-t_1)\delta c-(h-f-v)\,(t_3-t_2)\delta c+f(t_4-t_3)\delta c.$$

A stationary value of the transport cost, which can be shown to be a minimum, is obtained when the above expression is set equal to zero,

i.e. when
$$\frac{t_3-t_2}{t_4-t_1}=\frac{f}{h-v}$$

which states that the proportion of the time for which barges should be hired is $f/h-v$. Kirby's result corresponds to the case $v=0$.

The derivation given here makes it clear that the result is valid for any periodic pattern of demand caused, for example, by seasonal fluctuation. What is more, a steady trend can be superimposed on the demand curve without invalidating the result.

In the real life situation, barges are not of uniform size; hence f, v and h are not constants. However, the method of derivation implies that values of f, v and h applying to "borderline" barges should be used. From this a practical iterative method emerges in which the fleet size resulting from the use of average values for f, v and h is successively amended using "borderline" values. In fact, in the application of this formula to a case involving seasonal demand, the proportion $f/h-v$ proved to be $\frac{1}{2}$, thus implying that the fleet should be just big enough to meet the average demand.

Minimization of transport costs, given a certain demand for transport, represents only a part of the problem of minimizing costs in a distribution network. A further reduction in transport costs, without any change in fleet size, would be possible by oversupplying during periods of low demand using Company-owned barges and thus reducing the use of expensive hired barges during periods of high demand. This would necessitate increased storage capacity at the depots being supplied. Criteria are being developed to determine in what circumstances this would be advisable.

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IN an article entitled "A Multiple Re-order Point Inventory Policy", in the May 1961 issue of *Operational Research Quarterly* (Vol. 12, No. 1), E. Koenigsberg suggests a modification of the Wilson economic lot size formula, to take account of the inter-relationship between re-order level and re-order quantity. Assuming that the objective of an inventory control system is to provide a specified service (probability of stock shortage) at minimum cost, he attempts to evaluate the optimum re-order level and re-order quantity for the usual inventory cost model (stock holding cost proportional to average stock and ordering cost proportional to number of orders). The approach to the problem is interesting, but there are several points open to criticism. In Table 3 on page 39, for example, there are two raw materials for which the average stock assigned to the Wilson formula is greater than the re-order level plus the re-order quantity! In this letter, however, we are mainly concerned with commenting on the basic mathematics.

In the original paper, the method of ensuring constant service is to consider only policies with average stock equal to a constant k times the standard deviation of stock σ_M. Unfortunately, these policies do not always give the same service, as the following example shows.

An item has demand in the lead time distributed according to a negative exponential distribution with mean 400, and a protection level of $k = 2$ is specified. The table below contains ten possible policies. The average percentages of demands not satisfied immediately were calculated from the formula given later in this letter.

TEN POLICIES FOR ONE ITEM WITH $k = 2$

Re-order level R	Re-order quantity Q	Percentage of demands not met immediately	Re-order level R	Re-order quantity Q	Percentage of demands not met immediately
1152	100	4·95	971	600	4·55
1108	200	4·92	946	700	4·44
1068	300	4·86	923	800	4·29
1032	400	4·77	902	900	4·16
1000	500	4·64	887	1000	4·00

It can be seen that the percentages of stock shortages are not equal. The total cost of ordering and holding stock is minimized subject to a purely artificial constraint, and the results this method gives may well be different from those desired.

The formula for the percentage of stock shortages in a re-order level system with negative exponential demand in the lead time is particularly simple. Using Koenigsberg's notation[1] it is:

$$\text{Percentage of stock shortages} = \frac{100}{Q}\lambda_l \exp\left(-\frac{R}{\lambda_l}\right)\left[1-\exp\left(-\frac{Q}{\lambda_l}\right)\right].$$

The cost of ordering and holding stock,[2]

$$C\left(R+\frac{Q+1}{2}-\lambda_l\right)+\frac{S\lambda L}{Q},$$

can be minimized subject to the constraint that the percentage of stock shortages is a constant, θ say. The optimum Q is the solution of the equation.

$$\exp\left(-\frac{Q}{\lambda_l}\right) = \frac{(2S\lambda L/C)+2\lambda_l Q-Q^2}{(2S\lambda L/C)+2\lambda_l Q+Q^2}$$

and is, in fact, independent of θ. Taking a particular example where $\lambda_l = 400$, $C = 0{\cdot}010$, $S = 0{\cdot}5$, and $\lambda L = 2000$, this equation gives an optimum value for Q of 800. For the same values, Koenigsberg's formula with $k = 2$ gives $Q = 642$ and the Wilson formula gives $Q = 447$.

A further point on the paper is that it is clear from equation (11) on page 32 that Koenigsberg's optimum Q is less than Q^*, the Wilson economic lot size, when k is greater than

$$\sqrt{\left[3+\left(\frac{6\sigma}{Q^*}l\right)^2\right]}.$$

But it can be shown that the optimum Q is always greater than Q^* when stockholding and ordering costs are minimized subject to the constraint that the average number of shortages should be a fixed constant. Thus it seems that Koenigsberg's results are not valid for large values of k, and since he admits that his analysis is inaccurate for $k < 1{\cdot}5$, its range of applicability is at best very limited.

In some recent work on re-order level systems for inventory control of spare parts at the National Coal Board, the present writers also found the usual economic lot size formulation unsatisfactory, for reasons similar to Mr. Koenigsberg's. Our research followed a rather different line, however, since we estimated the cost of a shortage and developed a method of locating the re-order level and re-order quantity which gave minimum total cost.

We assumed captive demands, a known lead time l, and that the demand in time l was independent of previous demands. Using the approach of Galliher, Morse and Simonds we derived the following results:

$$\text{Average stock} = \sum_{y=0}^{R}\left(R+\frac{Q+1}{2}-y\right)d(y,l) + \frac{1}{2Q}\sum_{y=R+1}^{R+Q}(R+Q-y)(R+Q+1-y)\,d(y,l)$$

Average number of stock shortages per year =

$$\lambda\left(1-\sum_{y=0}^{R}d(y,l)-\frac{1}{Q}\sum_{y=R+1}^{R+Q}(R+Q-y)\,d(y,l)\right)$$

$$\text{Average back-up} = \text{Average stock}-\left(R+\frac{Q+1}{2}-\lambda_l\right)$$

Average number of orders per year $= \lambda/Q$.

Since these are all only single summations they lend themselves to manipulation on a digital computer, without the necessity of using simpler approximations.

The total cost function can be shown to have only one local minimum when the holding cost is proportional to the average stock, the shortage cost is a linear combination of the number of run-outs and the average back-up, and the ordering cost is proportional to the number of orders. The minimum was located in practice by an iterative procedure on a computer. Tables of the optimal values of re-order level and re-order quantity for different values of price and demand were calculated and are now in limited use. Values in the

tables differ markedly from the Wilson economic order quantity, particularly for high-priced items, for which high proportions of stock shortages are sometimes allowed.

We hope to publish these results in detail soon, since it seems probable that the tables, formulae and iterative procedure have wider application, provided there is some possibility of evaluating shortage costs. Even where this seems difficult, we feel some effort should be made to arrive at a reasonable estimate, since the acceptance of a standard shortage probability is always subjective, and is often irrational, when different items cost very different amounts to hold in stock. In practice it is not serious if the estimates of shortage costs are inaccurate since an estimate in error by a factor of as much as 2 will lead to a policy costing only a few per cent more than the optimum.

1. The symbols used throughout this letter are the same as those used in Mr. Koenigsberg's paper, but they are listed below to save repeated reference to the paper.

R denotes the re-order level
Q denotes the re-order quantity
l denotes the lead time
$d(y,l)$ denotes the probability of a demand y in time l
λ_l denotes the mean demand in the lead time
σ_l denotes the standard deviation of demand in the lead time
σ_M denotes the standard deviation of the inventory
L is an arbitrary time period
λ denotes the mean annual demand
S denotes the cost of an order
and C denotes the cost per unit of holding inventory over a time period L.

2. The formula

$$R+\frac{Q+1}{2}-\lambda_l$$

used for the average stock is not the exact average stock but is the approximation used by Koenigsberg. The exact formula is

$$\frac{R+Q}{2}-\lambda_l+\frac{\lambda_l^2}{Q}\exp\left(-\frac{R}{\lambda_l}\right)\left[1-\exp\left(-\frac{Q}{\lambda_l}\right)\right].$$

It is interesting to note that the optimum value of Q obtained using this formula is the same as that obtained using the approximation.

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Book Reviews

Sample Design in Business Research.

W. EDWARDS DEMING.

John Wiley, London. £4 16*s*.

In spite of its title, this book is not a reference book for an operation researcher going about his daily affairs. As the author writes at the start of his preface, "This is a text book", and by that he means a text book for students with many Examples and Exercises. It is written very explicitly and deals with experiences of sampling firms and human populations. The class of the problems handled, therefore, are few in number and one accumulates a feeling of disappointment at the poverty of the actual examples themselves. Not one is more intricate than selecting homes from an area without Census data, or sampling readers of a journal when they are only one in twelve of the population. Deming stresses the importance of replicated sampling, this might be said to be the central theme of the book, and he has a long chapter on the Theory of Variances.

Deming maintains that the statistician does not need to be an expert in the subject matter he is handling and it is abundantly clear that he is far more interested in the techniques of sampling than in the problems sampling is designed to answer. This attitude of his has dangers. Firstly, it makes it difficult for a statistician to contribute to the important problem of whether the right questions are being posed for examination. Secondly, it can lead to a lack of perspective verging on the ludicrous. For instance, after a great deal of computation on a sample of 1000 cases, Deming arrives at 2·3 per cent as the estimate of the 3-sigma limits of sampling variation, on the proportion of 73·7 per cent giving "no" as the answer to an opinion question. Sampling variation will not affect in the slightest management's decision when faced with such a majority, of three to one, yet Deming soberly quotes the client to the effect that narrower limits (than 2·3 per cent) would have been no more useful.

A final danger in Deming's attitude is one very germane to the whole problem of sampling. It is that he may be led to overlook the interviewing-situation involved in a particular survey. The sampling expert needs to know that certain subject matters cause an impossible situation if enquiries are made about them in the home. Let Deming try to ask about smoking or drinking in such circumstances and he will secure results of about 50 per cent of the true value. This is where research knowledge and skill come into play.

But let us list Deming's virtues, on which the book can be highly commended:

(a) Deming is crystal clear, with a style that is not tedious. American writers please copy.

(b) He has had wide practical experience and draws on this not only for the purpose of theory but to show how problems in the fieldwork can and should be tackled.

(c) He has a sane insistence on his belief that the place for arithmetic is before and not after the sample is drawn and the fieldwork done. Simplicity in the computation of results is the theme that he constantly reiterates.

HENRY DURANT

The Environs of Tagore (v + 135 *pp.*) and **Then and Now** (vii + 123 *pp.*).
HASHIM AMIR ALI.
Asian Publishing House and the Statistical Publishing Society, Calcutta, 1960.

These two books by Dr. Ali are part of a set of three studies dealing with a particular and limited development project in "rural reconstruction" in west Bengal. The third book, *The Bengal Peasant* (forthcoming) is to complete the series.

Because only two of the set are under review here it is difficult to assess the study as a whole or even deal with them as separate entities, for Dr. Ali insists that the three are complementary and therefore a "rounded picture" of the project cannot be obtained until the third book is available. However, the reader can obtain from these two books the approach and some of the results of the study. The material presented is only an analysis of statistical data and turns out to be a "pilot study" for a "pilot study" to be carried out in the same area in the near future.

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In *The Environs of Tagore*, Dr. Ali discusses the value of "Regional Surveys" as an essential preliminary to planning for rural development on any scale, and he suggests that for such applied research to be useful it must be confined to regional studies of considerable detail. But as he tells us in the second book (*Then and Now*, pp. 107 *et seq.*) the greatest difficulty lay in choosing a unit, not only for application of the methods of "rural reconstruction", but primarily for the purposes of a worth-while study upon which these methods could be based. The choice of the unit for the more extensive survey (*The Environs of Tagore*) is eventually made on geographical grounds as a result of some rather obscure reasoning, and the book presents the statistical data from a survey of an area known as a "Union" (a "self governing" unit of nearly 200 villages).

The material is given mainly in the form of charts and statistical maps, there being 43 of the latter. These are, on the whole, good but the insistence on explaining correlations by some very dubious geographical arguments rather detracts from the value of the information presented.

We are shown that for this small rural area of some 170 villages the main problems arising are: (a) a rapidly increasing population; (b) a limit to the expansion of cultivable areas; (c) an increasing dependence on land due to lack of other outlets and the increasing population density and (d) decreasing crop yields. The author deals with this situation in considerable statistical detail and reasonable argument when he is discussing the broader problems of population

increase and diminishing yields, but his more limited geographical interpretations are less convincing (e.g. p. 64).

Turning to the attempts at development which were initiated over 35 years ago Dr. Ali gives us a critical appraisal and suggests that little success was achieved because of the following factors: insufficient planning, overlapping of functions of various branches of the Visva-Bharati and the government teams in the area and the fact that the Development Block idea and the five-year plan are not sufficiently geared to the existing institutions and local conditions (pp. 58–59). He then proposes several useful studies which could be carried out, the priority problems to be faced being population growth (he suggests controlled male sterilization as an answer) and production increase. In this volume, caste is not dealt with as a factor in the economic system, but in *Then and Now* it is central to the analysis.

Then and Now is an analysis of detailed statistical data collected in five villages, all of them within two or three miles of Sriniketan, and Santiniketan, and is a comparison of two sets of figures, the first collected in 1933 and the second in 1956–8. Thus the author and his co-workers have been able to assess the impact existing development methods have had over the past 25 years in the context of specific village units, in contrast to the "regional analysis" of the previous study. The main points arising from this analysis are: that the "economic classes" based on "caste groups" have remained virtually unchanged over the past 25 years; population growth in the villages has been phenomenal; land ownership and land usage are still dependent almost entirely on caste affiliations.

Although Dr. Ali provides us with so much interesting information the studies are marred by lapses into completely subjective speculation and irritating vagueness. We are asked to consider that to show up lack of development in the area might ". . . reflect upon the fair name of Rabindranath" and that this produces a ". . . poignant problem" (p. 58). Also the words "household" and "family" are used throughout with little discussion of what is meant by them in this context, and there are some easily avoidable misprints in the text and elsewhere.

However difficult it is to maintain a reasonable level of objectivity in a study designed to fit a programme of rural development directed to certain ends, these two books give the reader the impression of being a "rushed job" over which insufficient care has been taken. It really is disconcerting to have the population density and the beauty of the scenery and sunsets described in the same section (*The Environs of Tagore*, pp. 17–18).

But with all this Dr. Ali and his assistants, in the extracts from their diaries written during the collecting of data in 1933, give us the impression of young men with tremendous enthusiasm and ideals who were trying to make their knowledge work in an area beset by problems of poverty and social inequality. Perhaps this should be the true spirit of "applied research".

P. J. A. RIGBY

Decision Models in Inventory Management.

R. B. FETTER and W. C. DALLECK.

Richard D. Irwin, Inc., Homewood, Ill., U.S.A., 1961. $5.75.

This work is divided into three chapters, "Inventory Decision Models", "Input Data" and "Numerical Examples", these together take up 86 of the 123 pages of text, the rest being Introduction, Summary and Appendices. It is well bound and produced in clear type commendably free from misprints.

It is a useful but at times an irritating book. Useful, because it brings together in a concise form work which has been scattered through various books and journals. Irritating, because of its errors and inconsistencies. For example, the partial differential equations on pages 15 and 17 need to have their right-hand sides divided by two—a regrettable error since the authors go on to stress, very properly, the importance of sensitivity tests. Continuous and discrete functions appear to be interchanged at the authors' whim. The preface states that "this material should serve as a guide for the interested and reasonably sophisticated operations analyst", and the text in general bears out this claim: why then on page 38 does it suddenly descend to a kindergarten discussion of variability?

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Having stated these criticisms—and the list could be extended—let me say that this book has proved its worth by being put to practical use even while under review, with potential borrowers held at bay. It deals in the main with four points:

(i) the interaction between re-order quantity and re-order level;
(ii) demand in a variable lead time;
(iii) problems of *inventory* rather than merely of *stock centres*: the effect of overall restrictions on groups of stocks is considered;
(iv) compiling input data.

Simulation and numerical methods are discussed; it is a pity that the authors do not at least mention the application of feed-back control theory which is arousing so much interest nowadays. In general, this book can be recommended, with some qualifications, to the serious practitioner in inventory control.

A. BATTERSBY

Review of Management International.

This journal is published jointly by The International University Contact for Management Education, Comité International de l'Organisation Scientifique, European Association of Management Training Centres, and The Institute of Management Sciences, and is an international review of management and managerial science. The first issue of this journal has just appeared and it proposes two reasons for producing an addition to the range of literature for management. The first is the need to speed up the interchange of management

ideas between those beginning to search for them; the second is that many specialist fields on which managers draw are not reaching the managers' attention because they are published in technical journals to which they normally have no access. Underlying all this is the desire to bring the practical manager and the academic more closely together. Each paper in the journal is printed four times, in English, French, German and Italian.

We have no doubt that the existing specialist management journals will be reviewing this new publication from the point of view of the manager. It is the purpose of this note to review the publication from the standpoint of the operational research worker and, hence, the journal is subjected to two tests:

(a) Does it tell the operational research worker of new and important developments in contiguous areas of which he should be aware?

(b) Is the journal, in the operational research man's experience, likely to encourage the spread of quantitative and analytical thought amongst managers?

One must beware of reading too much into too little and of criticizing a journal on the basis of its first issue. It must be said, however, that this first issue, which contains papers on The Function of Management, Science and the Manager, Investment Policy, Variance Analysis, Organization Charts and People, Discipline in Industry, and Trends in Marketing Studies, has a distressing variability in quality. It would perhaps be invidious to mention three papers in this journal, which, in the opinion of this reviewer, are naïve, and innocent to the extreme. They are all liable to do the cause of the research worker in industry some disservice by their evident surprise that the hardened experienced industrialist does not welcome with open arms the bright-eyed research student and by their lack of knowledge of the business scene. On the other hand, two of these papers are well worthy of study by the operational research worker. The first is a useful, if brief, survey of the Function of Management by Sir Frederic Hooper. The second is a paper on Flexible Budgets and the Analysis of Overhead Variances by Professor D. Solomons. This paper gives a good exposition on the way the accountant regards some fairly simple costs situations. The fact that it poses a number of questions to the operational research worker adds to its value.

On the very limited basis of this first number there are papers in this publication which will be of value to the operational research worker if he is prepared to sift and sort out the pearls. (This sorting is made difficult by the multiplication of languages and four separate journals would be a distinct improvement.)

We feel, however, that the presence of some papers which are trivial and academic (in the pejorative sense of the word) will harm the cause of quantitative analytical thinking in management science. The busy manager is more likely to be put off by the trivial than he is to be attracted by the important.

B. H. P. RIVETT

Notes on Operations Research. Operations Research Centre, M.I.T.

This book, mainly by staff of M.I.T., is based on the summer courses on operational research presented there over the last few years. Except for the introduction, which stresses the need for practical as well as theoretical knowledge of the subject (but unfortunately gives the impression that operational research is exclusively concerned with stochastic systems), the book is mainly mathematical.

It does not pretend to cover all aspects of operational research (there are notable exceptions in Game Theory and Inventory Systems), but sets out to present recent theoretical advances in operational research. While four of the chapters can be classified as such, the remainder of the book is concerned with well-known techniques and areas of application and the preface is a little misleading in this respect. Although this familiar material is mostly competently presented, the reader hoping for new techniques on maintenance problems, for example, is likely to be disappointed.

Nevertheless, the book presents some novel ways of tackling various problem areas, and some very specialized theory on some aspects.

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The authors admit that continuity and consistency have been sacrificed for speed of publication and while this is laudable to some extent, the shortcomings are apparent in some instances. The most curious example is the inclusion of a very short chapter on the organization of operational research groups, which seems out of place in a book otherwise mainly devoted to mathematical techniques. Elsewhere this inconsistency is noticeable in the different methods of presentation used in the separate chapters, and in the different levels of mathematics at which they are written—some chapters leading the reader gently from first principles, and others demanding a considerable mathematical background.

Considering the separate sections of the book, the new material is almost wholly confined to the chapters on Markov Processes, Control Processes, Sequential Decision Theory, and Information Theory respectively. These topics are well presented, giving new theoretical treatments and suggesting techniques which might be applied in certain problem areas, but the mathematical content is necessarily formidable.

The remaining chapters cover, some more successfully than others, well-known techniques and areas of application.

Chapter 1, on probability theory, starts from first principles and gives several applications, while Chapter 2 is a very clear and concise presentation of many aspects of search theory—an unusual inclusion.

Chapter 4 gives a summary of theoretical progress to date in queueing theory, and some analytical development, and Chapter 8 is a simple but clear account of some maintenance problems and the methods available to handle them.

Production Scheduling is considered in Chapter 9, though in something of a piecemeal manner. After an indication of the use of linear programming in

scheduling problems, the application of dynamic programming to problems of decaying demand is considered, and finally the effects of probabilistic demand on scheduling policy are briefly discussed.

The final chapter considers one or two aspects of simulation, mainly from the view of computer application, and indicates sampling techniques for several types of distribution.

Unfortunately there are many errors and omissions throughout the text, and these and the other signs of hurried preparation do detract somewhat from its merits. However, as some compensation for this irritation, it sells at a very reasonable price, and the offset printing is very clear.

In summary, then, while regretting that some of the chapters fall well below the standard in content and presentation reached by the better ones, these latter alone make the book good value for money.

M. G. SIMPSON

Operations Research. E.P.A. Project 6/02A–1 Final Report.

Organization for European Economic Co-operation, European Productivity Agency. Paris. March, 1961. 18 *pp.*

This is the final report of Dr. R. W. Shephard on completion of his tour of Europe as consultant on this E.P.A. project which is stated to be "the initial effort of E.P.A. in the area of operations research". The purpose of the project was to conduct seminars and courses on operations research in member countries of O.E.E.C. with wide objectives for promoting the application of the subject and assisting in setting up courses and industrial groups.

The report contains an introductory section, four pages devoted to a description of Dr. Shephard's visit to nine European countries, four pages of observations, a page of recommendations, two pages giving his *curriculum vitae*, and two giving an outline curriculum for an operations research seminar. No doubt as a report to E.P.A. this gives an accurate description of Dr. Shephard's activities, and he is fully entitled to his personal views on some of the controversial subjects which he picks on for mention in his observations. However, for the sake of those who may come upon the report and, seeing the stamp of E.P.A. upon it, feel it has some form of official sanction, the following comments must be made. The views which Dr. Shephard expresses are his own and many would find little support among the great body of operational research workers. The reason for this appears to be partly because Dr. Shephard's concept of operational research itself is, one infers, a restricted one. No doubt this arises from his own background which appears mainly to be in industrial engineering, although the position is made obscure by the fact that in one place he is described as now being Professor of Engineering Science at the University of California in Berkeley and in another Professor of Industrial Engineering at the same address; it is perhaps significant that the list of his professional

affiliations does not include the Operations Research Society of America. A second reason for accepting his observations with reserve lies in the way they were made. The reviewer can only speak of the position in the U.K., but so far as that is concerned there is no doubt that the institutes which Dr. Shephard visited there would be the last to claim that they were leaders in the advance of operational research. He appears to have visited none of the flourishing industrial groups in the U.K. and if he was kept as well on the fringe of the subject in the other countries he visited it is no wonder that he can write: "The subject matter of operations research is highly technical with a strong requirement for background in mathematics and statistics. In the majority of countries visited it was found that this technical background was deficient."

The interest of E.P.A. in this subject is much to be welcomed and they are to be congratulated in having promoted this project. It is hoped, however, that this report is nowhere accepted uncritically as an authoritative expression either of fact or of opinions other than the author's.

R. T. EDDISON

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Abstracts

Predicting Distributions of Staff.

YOUNG and ALMOND.

The Computer Journal, **3**, 4, pp. 246–250 (January, 1961).

A method is described which was used in an organization to predict the effect of continuing the present policies of recruitment and promotion upon the probable numbers of staff among various grades in future years.

B. H. M.

Queueing with Balking, Part 2.

F. A. HAIGHT.

Biometrika, **47**, 3 and 4, p. 285 (1960).

The queueing model considered is that in which arrivals are Poisson and the service time negative exponentially distributed, but with some arrivals balking if the queue is large. This critical value of the queue length is a random variable with a general distribution. Queueing with balking has been considered in various forms by several people including Haight himself in an earlier paper (*Biometrika*, **44,** 360 (1957). This paper deals particularly with the join-balk sequence, and the distribution of the gap times between joining arrivals is obtained. The reason for finding this distribution is that in the traffic problems which lead to the paper, the balking can take place at some distance from the congestion, and hence not be observed. In practice, the observed interjoining gap distribution is compared with the negative exponential and with the theoretical distribution obtained in the paper.

The paper closes with a numerical illustration comparing the interjoining gap distribution of a simulated queue with a Poisson balking rule with some standard distributions.

R. W.

Self-optimizing Control Systems.

A. P. ROBERTS.

Control, **4**, 34, pp. 84–89 (April, 1961).

Extreme Regulation, Invariant-Response Control and Input Adaptation are introduced as different types of self-optimizing systems. A brief qualitative discussion of these which mentions the use of cross-correlation in control systems leads to a short treatment of Input Adaptation. This arises when a system has to adapt its own control characteristics to achieve an overall optimum behaviour

in the face of some arbitrariness in the input information. The problem is approached in two general ways, the first being an extension of Wiener's theory, the second Dynamic Programming. The latter is becoming an important synthesis procedure for self-optimizing control systems.

W. M. D.

Stock Control Problems. Some fallacies in their current treatment.

E. DUCKWORTH.

Journal of Applied Statistics, **ix**, 3, pp. 133–151 (November, 1960).

It is argued that published rules for determining buffer stocks and re-order quantities are inappropriate. In practice they would lead to the holding of excessive stocks. These rules do not take account of the variation in delivery time. They also equate the proportion of those commodities which will be out of stock at a given time with the combined probability of a "stock-out".

Formulae are derived for the proportion of ordered items out of stock before replenishment, the proportion of all items out of stock and the length of time out of stock. The results are checked by simulation.

The size of an individual order is often chosen to minimize the total cost per piece using the "square root" formula. This is not necessarily a sound method for a firm with limited capital. The implications of the "acceleration effect" for the size of buffer stock are also discussed. The principal recommendation is that stock control decisions should be made at Managing Director level. The industrial statistician's task is then to devise simple and consistent schemes providing the required level of protection.

D. J. B.

Theory of Games and its Application to Rate of Interest.

S. BENJAMIN.

Journal of the Institute of Actuaries, **85**, 371, pp. 373–437 (December, 1959).

A games model is used to show the way the working of a Life Insurance Office is affected by the changes in the market rate of interest. A concept of an "average rate" over a period of time is shown to be unsound and the problem of investing to the best advantage is reduced to a two-person zero-sum game between the investor and the "market".

J. H. C.

Responsive Systems—a note on individual cybernetics.

S. BEER.

F.B.I. Review, pp. 39–41 (March, 1961).

A short descriptive article. Cybernetics is described as "the science of control", with particular reference to the relation between information flow and control

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in a system, the effects of redundancy of information and of channels, and the possibilities of systems that can recognize patterns, learn by experience, and modify their performance in the light of changing internal and external circumstances. The application of these ideas to the control of plant, to the planning of operations, and to the improvement of organization is indicated.

T. E. E.

The Use of Computers by the Oil Industry.

H. A. LEACH.

The Petroleum Times, **65**, 1569, pp. 191–193 (10 March, 1961).

This article describes in general terms three fields in which computers are being used by the oil industry, namely in refinery planning and control, engineering analysis and design, and research. A brief example of each is given together with observations on desirable machine characteristics and programming systems.

H. R. W. W.

Operational Research in London Transport.

A. T. WILFORD.

British Transport Review, **vi. 2**, pp. 124–134 (December, 1960).

A brief description is given of the unit and its methods of operation. Most of the paper describes a wide range of bus and underground investigations. These include some ergonomic aspects of vehicle design, methods of reducing diesel fuel consumption and the performance of buses when affected by road traffic congestion in Central London. As well as the usual type of investigation into the optimum use of existing facilities, investigations have been carried out when the work being studied was in the planning or prototype stage.

P. F.

Techniques for Producing School Time-tables on a computer and their application to other scheduling problems.

APPLEBY, BLAKE and NEWMAN.

The Computer Journal, **3**, 4, pp. 237–245 (January, 1961).

The progress made in devising a technique for drawing up school time-tables using a computer is described, and details of three programs are given. The relation between this work and the general problem of scheduling is briefly discussed.

B. H. M.

Computer Aids for Air-traffic Control.

ANON.

Engineering, **191**, 4955, p. 479 (7 April, 1961).

After a brief survey of British thought in this field there follows a description of experimental results obtained at Indianapolis, U.S.A. There, a standard computer, the I.B.M. 650 with a R.A.M.A.C. file, has for the past three years been programmed to process flight-control data to aid the flight controllers.

Information by teletype, telephone or voice-transmission is fed to the computer automatically or by hand punching of cards as appropriate.

The computer processes and stores flight-plans and flight-data to obtain actual and predicted flight-progress. Calculation of predicted progress includes allowance for wind.

Information included in the flight-plan is flight-identification, aircraft type, airspeed altitude, point of departure, route destination, etc. Flight-data contain fixes, windspeed, etc.

When flight-progress has been calculated from these, it is printed out, either actual or predicted or both, in a form for visual use by the controllers.

The computer mentioned takes 10 seconds to process a flight-plan.

M. W. B. T.

Dynamics of Halting a Line of Traffic.

W. A. WOODS.

Engineering, **191**, 4960, pp. 655–656 (12 May, 1961).

On the basis of the similarity between the transient flow of gases in pipes and the unsteady flow of vehicles through the Mersey Tunnel, the fundamental principles of dynamics are applied to the traffic problem. Using simple dynamic theory, the thinking distance and the braking distance which together comprise the stopping distance are defined in terms of vehicle speed. From the minimum safe spacing of a one-lane traffic line, the maximum flow rate is determined as a function of speed and vehicle length.

P. H. A.

A Predictive Model of Self Organizing Systems.

G. PASK and HEINZ VON FOERSTER.

Cybernetica, **iii**, 4, pp. 260–300 (1960).

The application of game theory to systems which exhibit competition and co-operation is discussed. The object is to clarify the game theoretic description of such systems and so permit, in some cases, the prediction and evaluation of methods of play. Self-organizing systems are discussed with reference to current experimental work. The first experiment is one in which human subjects

participate in a "social interaction system". The subjects, who may form coalitions, compete for wealth in a system in which channels of communication of money and information are changed by use. The game theoretic description is developed. The second part of the paper will appear in the next issue of the journal. It will describe the experiment in which the human subjects in the social interaction system are replaced by automats.

D. A. H.

Operational Research—Its methods and applications.

A. H. GOULD.

Journal of the Institute of Actuaries, **86**, 2, p. 373 (1960).

Defines and introduces operational research as a means of providing scientific assistance at the policy-making level, using methods derived from different scientific fields. It gives an outline of the methods employed, with special reference to the similarity of the treatment of certain problems with existing actuarial techniques. Describes the future applications in the armed services and in industrial and commercial fields and the training of the operational research scientists.

Appendices describe the history of operational research and some problems solved.

J. H. C.

News and Notes

New Full Members of the Society

N. A. M. Eastwood—Rio Tinto Management Services Ltd, London.

R. W. Elliott—National Coal Board, London.

G. W. H. Field—The United Steel Companies Ltd., Sheffield.

A. H. R. Hurt—Société de Recherche Opérationnelle et d'Economie Appliquée, Brussels.

L. F. Jones—Armament Research & Development Establishment, Fort Halstead.

Dr. L. B. Kahn—Shell Development Company, Emeryville, California.

J. S. McBride—Williams & Williams Ltd., Chester.

A. Muir—The United Steel Companies Ltd., Sheffield.

R. C. Tomlinson—National Coal Board, London.

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Changes of Appointment

P. G. Barnes, previously with de Havilland Aircraft Co. Ltd., is now with C.E.I.R. (U.K.) Ltd., London.

R. W. Bevan, O.B.E., previously at the Air Ministry, is now with Headquarters Fighter Command, Stanmore.

A. D. Cunningham, previously with B.I.S.R.A., is now at Cambridge University, Department of Engineering.

N. A. M. Eastwood, previously with Rio Tinto Management Services Ltd., is now with A.E.I. Hotpoint Ltd., London.

M. G. Ferrand, previously with E.M.I. Electronics Ltd., is now with Massey-Ferguson (UK) Ltd., Coventry.

A. D. J. Flowerdew, previously with the National Coal Board, is now with Richard Thomas & Baldwins Ltd., Newport.

R. J. Griffiths, previously with Metropolitan Vickers Electrical Co. Ltd., is now with British European Airways, Ruislip.

F. de P. Hanika, previously at the Royal College of Science & Technology, is now at Churchill College, Cambridge.

K. F. Lane, previously with The Carborundum Company, is now with Rio Tinto Co. Ltd., London.

A. M. Lee, previously with British European Airways, is now with Trans-Canada Airlines, Montreal.

Dr. J. G. Laski, previously with The United Steel Companies Ltd., is now with Esso Petroleum Co. Ltd., London.

Dr. J. R. Lawrence, previously with the National Coal Board, is now with Richard Thomas & Baldwins Ltd., Ebbw Vale.

Dr. E. F. K. Liebel, previously with Bergbau A.G. Ewald Koenig Ludwig, is now with Aschaffenburger Zellstoffwerke A.G. Redenfelden.

J. H. Nicol, previously with the National Coal Board, is now with Arthur Andersen & Co., London.

National Conference, 1961

The Society's national conference will be held from 24th to 26th October at Leamington Spa.

The programme will include major review sessions, the papers in which will consider critically some of the techniques of operational research. The following discussion groups are also planned:

Factors Affecting the Implementation of Results.
The Effects of Automation on Management Structure.
Problems in the Determination and Allocation of Costs.
Mental Approaches as Part of the Operational Research Technique.
Operational Research at the Strategic Level in Industry.
What do the Members Expect from the Society?

The opening address will be given by Professor R. L. Ackoff, Case Institute of Technology (Joseph Lucas visiting professor in the Department of Engineering Production at Birmingham University), and the closing address will be given by Mr. Stafford Beer, The United Steel Companies Ltd.

On the Wednesday evening there will be a Banquet which will be attended by the President of the Society, the Earl of Halsbury, who will give an address.

Full details and application forms can be obtained from the Secretary of the Operational Research Society, 64 Cannon Street, London, E.C.4.

Operations Research Society of America

The First Joint National Meeting of the Operations Research Society of America and the Institute of Management Sciences will be held at the Jack Tar Hotel, San Francisco, California, from 8th to 10th November, 1961.

ANNOUNCEMENT OF PUBLICATION

The Operations Research Society of America and the International Federation of Operations Research Societies announce the publication of a new quarterly journal, *International Abstracts in Operations Research*. The first issue is scheduled to appear in October, 1961. The journal will, on a current basis, cover operations research literature published in 1961 and subsequent years. The annual subscription price will be $12.50 in the United States and Canada and $10 elsewhere. Orders for subscriptions should be sent to the Business Manager,

Operations Research Society, Mt. Royal and Guilford Avenues, Baltimore, Maryland, U.S.A.

A professional venture. Abstracts will be produced by reviewers in the 12 member societies of the Federation, each society of which will cover the literature published in its own country, and by societies and reviewers in a number of additional countries. Among the countries whose literature will be covered by these abstracts are Australia, Belgium, Canada, France, Germany, Great Britain, India, Italy, Japan, Netherlands, Norway, Russia, Sweden, Switzerland and the United States. The current membership of the member societies of the Federation is in excess of 5000 individuals around the world.

Coverage. In each country a large number of journals will be scanned for articles of interest. For example, in the United States some 88 journals will be covered including all the principal journals in the following fields: Business and Industry, Computers, Economics, Engineering, Management and Operations Research, Mathematics, the Natural Sciences, Psychology, Sociology, Philosophy, and Statistics.

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English abstracts. The abstracts will be written in English, although of course many of the articles abstracted are written in other languages. Full source bibliographies will be provided.

A new kind of subject-matter index. In addition to an author index the journal will feature a novel subject index based upon "index titles" prepared by the reviewers who produce the abstracts. This subject-matter index will occupy approximately 15 per cent of the space in each issue. A sample of this index is shown below.

Circulation. The journal will have an extensive circulation through the participating societies to their members in the many countries represented.

Frequent, prompt publication. An essential objective of the journal is to achieve a very short time lag between original publication of the article and the publication of an abstract of it. To this end publication will be advanced from quarterly to bi-monthly as soon as practicable.

Editors. The board of editors will include contributing editors appointed by each of the member societies of the Federation. Among these editors are Herbert P. Galliher (United States), K. B. Haley (Great Britain), Jacques Agard and Gabriel Faiveley (France), H. C. Hearne, Jr. (Canada), Bruce Craven (Australia), J. Wanty (Belgium), Per Johannessen (Norway), S. Danielson (Sweden), J. J. Meinardi (Netherlands), Tatsuo Kawata (Japan), H. Kregeloh (Germany). The editor-in-chief is Herbert P. Galliher.

Subscriptions. There is enclosed order blank and return envelope. Inquiries concerning subscriptions and business matters should be addressed to the Business Manager, Operations Research Society, Mt. Royal and Guilford Avenues, Baltimore, Maryland, U.S.A. Inquiries concerning editorial matters should be addressed to Herbert P. Galliher, Operations Research Center, Room 6–218, Massachusetts Institute of Technology, Cambridge 39, Massachusetts,

U.S.A. (The new full subscription for all grades of members of the Operational Research Society Limited includes a subscription to *International Abstracts in Operations Research*—Ed.)

Organization of each issue. Each issue will begin with a subject index, described below, followed by 6 sections in which abstracts have been grouped, and terminate with an author index. The 6 sections are:

Section	*Comprising literature primarily concerned with*
Models of Common Processes	construction and solution of models of functions or activities common to enterprises in various arenas, including for example: Control, Forecasting, Information, Marketing, Scheduling
Experiment and Special Applications	experiments or applications in such arenas as the following, or models which are quite specialized to the arena: Commerce, Communication, Government, Manufacturing, Military, Transportation, etc.
Theoretical	abstract models and methodology, papers of primarily mathematical content and solutions of mathematical problems arising in abstract models, including: mathematical programming (Assignment, Networks, Search, etc.), probabilistic models (Monte Carlo, Statistics, Stochastic Processes, etc.), other mathematical and scientific topics (Chemistry, Combinatorial Analysis, Physics, Servomechanics, etc.)
Professional	the practice and development of operations research as a profession in itself
Books	
Bibliography	

SAMPLE OF THE SUBJECT-MATTER INDEX

Each line in the sample refers to a single abstract. The first column describes the principal results obtained or topics treated, the second column describes special aspects of the result or topic, the third column explains the technical methodology employed. The fourth column notes excellent aspects of the article abstracted, such as B (bibliography), C (computational aids), E (experimental), H (history of applications), M (model construction), R (review of methodology), T (theoretical). The fifth column gives the accession number (page reference in issue) of the abstract.

MODELS OF COMMON PROCESSES

Growth			
stochastic investing	gasoline marketing	dynamic programming	E 0001
Inventory			
Poisson, independent	replenishment	validity	0005
Marketing			
customer behavior	seasonal textiles	statistics	E 0007

EXPERIMENT AND SPECIAL APPLICATIONS

Commerce			
see also: 0001, 0006, 0007 profit-sharing	company staff	equations	CM 0018
Military			
see also: 0055			
target attrition	air defense	particle diffusion	MT 0027
Transportation			
driver-delays	Poisson traffic	probability	ME 0035

THEORETICAL

Dynamic Programming			
see also: 0002, 0006			
n parts, *m* machines	least time	graph	CM 0038

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Announcement

INTERNATIONAL FEDERATION OF OPERATIONAL RESEARCH SOCIETIES. THE THIRD INTERNATIONAL CONFERENCE ON OPERATIONAL RESEARCH

AT THE invitation of the Norwegian Operations Research Society, the Third Conference of the International Federation of Operational Research Societies will take place at the University of Oslo in July, 1963, from the 1st to the 5th inclusive. The programme committee invites suggestions as to papers and subjects which might be included in this Conference. Any such proposals should be sent to the Secretary of the local National Operational Research Society before 1 December, 1961.

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A Linear Risk Model†

WILLIAM S. JEWELL

Operations Research Center, University of California, Berkeley

A special structure optimization model is presented which includes many of the single variable risk problems that are encountered in operational problems. A risk function is assumed which is a piece-wise linear function of some random variable whose distribution is known; one seeks the value of the decision variable which minimizes expected risk. In this paper are presented the necessary and sufficient conditions for this optimization for random variables which are either continuously or discretely distributed. The important special case of a continuous risk function is discussed; multiple risk problems with a joint constraint are analyzed; and the change in policy for a small change in the distribution of the random variable is investigated. Examples illustrate the application of the model.

INTRODUCTION

CONSIDER the following simple optimization problems:

I. A newsboy pays C cents for each of the papers he orders from the distributor, and receives S cents from every customer who buys a paper. Customer demand follows a known distribution each day; how many papers should the newsboy order each day to maximize his profit over many days' operation? [5]

II. Exploration for ore deposits begins at a location estimated from geologic data, aerial photographs, etc. However, the actual location of the deposit has some radial error distribution about this initial point, which can be estimated from past experience. Exploration costs are proportional to the area explored. If the value of the deposit can be accurately estimated in advance, how large should the exploration radius be to maximize profits over many such explorations?

III. Bar stock is inventoried in discrete sizes; if an odd length is needed, it is cut from the next largest size, with the scrap being salvaged. Suppose that, because of a large demand for a range of non-standard lengths, it is desired to establish a new intermediate length between two original sizes. What choice of intermediate length will minimize the average net cost of operations? [1,3]

† This research was supported in part by the Office of Naval Research under contract Nonr-222(83) with the University of California. Reproduction, in whole or in part, is permitted for any purpose of the United States Government.

These problems are typical of the decision problems that are often encountered in operational analyses.[4, 5] A single decision variable, z, is to be chosen to optimize system performance in the face of an uncertain parameter, x. System performance is measured in terms of some risk function, $R(z; x)$; in most industrial problems, the risk will be measured in operating costs, or cost rate. If the distribution, $P(x)$, of the parameter x is assumed known, then optimum performance can be equated with minimum average risk for most situations where many trials are to be made. The problem then becomes an optimization problem in which we seek the value of the decision variable, $z = z^*$, such that $R(z) = \int R(z; x)\, dP(x)$ is minimized.

In this paper we shall investigate a class of such simple risk problems where the risk function has a special linear structure. Because of this special structure, the optimum solution has an extremely simple algorithm for both continuous and discrete distributions. In addition, certain important extensions retain the simplicity of the optimal solution.

Application of the linear risk model will be demonstrated by solving the problems just presented.

THE LINEAR RISK MODEL

The special risk function to be assumed is shown in Figure 1. The decision variable, z, is measured in the same units as the random parameter, x, and the risk function, $R(z; x)$, is assumed to be piecewise linear in two parts, depending upon whether or not x is less than z. Furthermore, the initial ordinate, $A(z)$, and the discontinuity, $D(z)$, are assumed to be functions of the decision variable, while the slopes of the risk curve are not.

Thus, the risk function is:

$$\begin{aligned} R(z; x) &= A(z) + m_1 x && (0 \leqslant x < z) \\ &= A(z) + m_1 z + D(z) + m_2(x - z) && (z \leqslant x < \infty), \end{aligned} \tag{1}$$

and we seek:

$$R(z^*) = \min_z R(z) = \min_z \int_0^\infty R(z; x)\, dP(x). \tag{2}$$

Instead of using the known distribution function $P(x)$, we shall find it easier to use the so-called tail distribution function, or complementary distribution function, $Q(x) = Pr(\geqslant x)$. By assuming that the first moment, $\bar{x}$, exists, one can use the transformations (derived through integration by parts):

$$\int_0^z x\, dP(x) = -zQ(z) + \int_0^z Q(x)\, dx, \tag{3}$$

$$\int_z^\infty (x - z)\, dP(x) = \bar{x} - \int_0^z Q(x)\, dx, \tag{4}$$

and

$$\bar{x} = \int_0^\infty Q(x)\,dx, \tag{5}$$

to find the following simplified expression for average risk:

$$R(z) = m_2\bar{x} + A(z) + D(z)\,Q(z) - (m_2 - m_1)\int_0^z Q(x)\,dx. \tag{6}$$

The optimization of this function will provide the best value of the decision variable, in the sense of minimizing the average risk for the risk function of Figure 1. Notice that this optimization is not affected by the addition of any risk function depending only upon x.

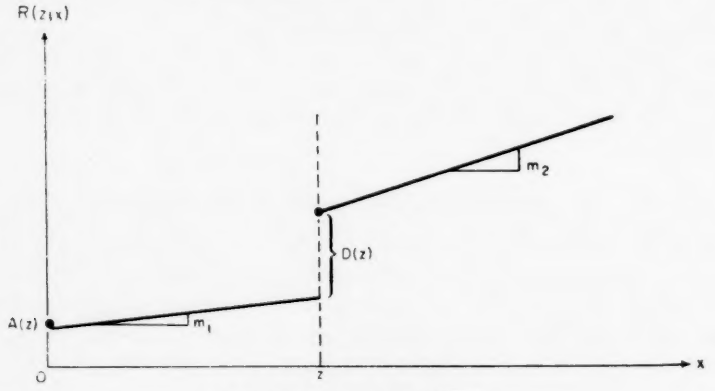

FIG. 1. *Linear risk function.*

CONTINUOUSLY DISTRIBUTED PARAMETER

If the random parameter is continuously distributed, then the tail distribution function possesses a derivative, $p(x) = -dQ(x)/dx$, which is the usual density function. Assuming that $D(z)$ and $A(z)$ have the necessary derivatives, we find that the necessary condition for an internal optimum at $z = z^*$ $(0 < z^* < \infty)$ is:†

$$R'(z^*) = 0,$$

or
$$[(m_2 - m_1) - D'(z^*)]\,Q(z) = A'(z^*) - D(z)\,p(z). \tag{7}$$

A sufficient condition that this point is a minimum is:

$$R''(z^*) > 0,$$

or
$$A''(z^*) - D(z^*)\,p'(z^*) + [-2D'(z^*) + m_2 - m_1]\,p(z^*) + D''(z^*)\,Q(z^*) > 0. \tag{8}$$

† We shall use the notation

$$R'(z^*) = \left.\frac{dR(z)}{dz}\right|_{z=z^*}, \quad R''(z^*) = \left.\frac{d^2R(z)}{dz^2}\right|_{z=z^*} \text{ and so forth.}$$

In many operational problems of the kind considered here, one can make the simplification that $A(z)$ and $D(z)$ are linear functions of the decision variable, i.e.:

$$A(z) = a_0 + a_1 z, \quad D(z) = d_0 + d_1 z. \tag{9}$$

In this case, the necessary condition reduces to:

$$(m_2 - m_1 - d_1)\, Q(z^*) = a_1 - D(z^*)\, p(z^*), \tag{10}$$

with a sufficient condition for a minimum of:

$$(m_2 - m_1 - 2d_1)\, p(z^*) > D(z^*)\, p'(z). \tag{11}$$

The computational program suggested by equation (10) involves finding the intersection of the tail distribution curve with the density curve which has been multiplied by a linear function of z. Once the analytic form of the distribution is known, this procedure can usually be carried out very quickly, as well as checking for a true minimum with inequality (11).

If several minima result, then the risk function of equation (6) must be used to separate out the overall minimum. In addition, one should use equation (6) to check that the optimal solution does not occur on a boundary point, z equals zero or infinity.

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DISCRETELY DISTRIBUTED PARAMETER

If x is a discrete random variable with $\text{Prob}(x = n) = p_n$ $(n = 0, 1, 2, \ldots)$, then $Q(x)$ is a staircase-like function, whose value at x equal to an integer is just $Q(x = n) = p_n + p_{n+1} + p_{n+2} + \ldots$. Since the function $Q(x)$ then drops immediately to the next value for x infinitesimally greater than an integer, we have the equivalence

$$\int_0^z Q(x)\, dx = \sum_{n=1}^{z} Q(n) \tag{12}$$

for z equal to an integer. The expected risk becomes:

$$R(z) = m_2 \bar{x} + A(z) + D(z)\, Q(z) - (m_2 - m_1) \sum_{n=1}^{z} Q(n), \tag{13}$$

where for the rest of the discussion we shall assume that we seek an integer value of z, since it is measured in the same units as the random variable.

By using the method of differences, we find that the conditions for a minimum risk at the (integer) point $z = z^*$ are:

$$\begin{aligned} &[m_2 - m_1 + D(z^* - 1) - D(z^*)]\, Q(z^*) > A(z^*) - A(z^* - 1) - D(z^*)\, p_{z^*-1} \\ &[m_2 - m_1 + D(z^*) - D(z^* + 1)]\, Q(z^* + 1) \leqslant A(z^* + 1) - A(z^*) - D(z^*)\, p_{z^*}. \end{aligned} \tag{14}$$

The equality allows for the possibility that there may be several adjacent minima; conditions (14) then select the smallest of the possible z^*.

If the linear simplification of equation (9) is valid, we have the simple tests:

$$\begin{aligned} &(m_2 - m_1 - d_1)\, Q(z^*) > a_1 - D(z^*)\, p_{z^*-1} \\ &(m_2 - m_1 - d_1)\, Q(z^* + 1) \leqslant a_1 - D(z^*)\, p_{z^*}. \end{aligned} \tag{15}$$

In either event, the computation program suggested is a quite straightforward one, once the tabulated values of the discrete distribution and its tail sums are available.

It is also possible to analyse the case of a mixed distribution for the random variable, but since the results are of little practical importance, they will not be reproduced here.

CONTINUOUS RISK FUNCTIONS

An important special case occurs when there is no discontinuity in risk when the random variable just exceeds the value of the control variable (for

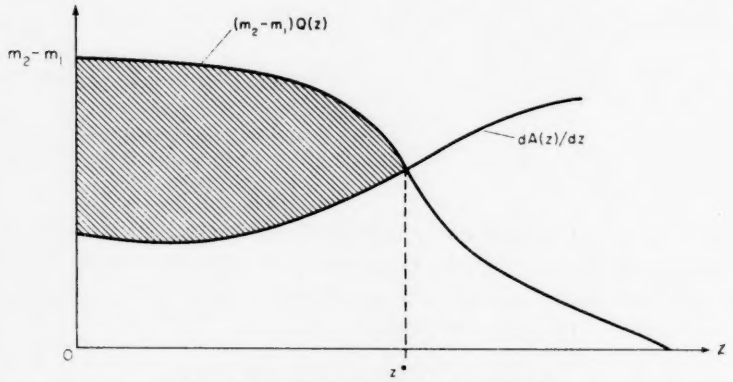

FIG. 2. *Graphical solution for minimal risk.*

instance, in the first example). In this case, $D(z) = 0$ for all z, and the risk function $R(z; x)$ is continuous in x. We find the criteria for optimality to be:

$$(m_2-m_1)\,Q(z^*) = A'(z^*)$$
$$A''(z^*)+(m_2-m_1)\,p(z^*)>0 \tag{16}$$

for the continuous case, and for the discrete case:

$$(m_2-m_1)\,Q(z^*)>A(z^*)-A(z^*-1)$$
$$(m_2-m_1)\,Q(z^*+1)\leqslant A(z^*+1)-A(z^*). \tag{17}$$

Equation (16) suggests the following interesting graphical construction, if there is a unique internal minimum. Plot $(m_2-m_1)\,Q(z)$ and $dA(z)/dz$ as functions of z on the same graph (Figure 2), and read off their intersection $z = z^*$ directly. One can easily show that the minimal risk is then:

$$R(z^*) = m_2\bar{x}+A(0)-[\text{Shaded Area in Figure 2}]. \tag{18}$$

A similar result obtains for discretely distributed x.

MANY VARIABLES WITH CONSTRAINT

When a limited common resource is shared among several risk situations, one may use N risk models of the kind described, together with the constraint

$$\sum_{i=1}^{N} \alpha_i z_i \leqslant M. \tag{19}$$

(We use the index i to denote each separate risk problem, $i = 1, 2, \ldots, N$.)

We will assume that total risk is just the sum of the individual risks. Probably the most efficient procedure for solving this problem is first to minimize each sub-problem separately, and check to see if the constraint (19) is violated. If it is not violated, then the multi-variable optimum is just the optimum of each sub-problem, since the total expected risk is just the sum of the individual expected risks, whether or not the random variables are correlated

$$R(z_1, z_2, \ldots, z_N) = \int\int\int\int R(z_1, z_2, \ldots, z_N; x_1, x_2, \ldots, x_N)\, dP(x_1, x_2, \ldots, x_N)$$

$$= \int\int\int\int \sum_{i=1}^{N} R(z_i; x_i)\, dP(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} R(z_i; x_i)\, dP(x_i). \tag{20}$$

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If the constraint is violated, and the individual risk functions are convex in the z_i, then one may show that the constrained optimum must occur on the boundary of (19), and the method of Lagrange multipliers may be used. One seeks the stationary points of the new function:

$$S(z_1, z_2, \ldots, z_N, \lambda) = R(z_1, z_2, \ldots, z_N) + \lambda\left(\sum_{i=1}^{N} \alpha_i z_i - M\right). \tag{21}$$

The entire analysis developed previously applies, except that wherever $A(z)$ and its derivatives occur, we substitute $A_i(z_i) + \lambda\alpha_i z_i$, together with the corresponding values of the other parameters for this particular sub-problem.

The Lagrange multiplier, λ, will of course be an unknown. However, because of the extremely simple way in which it enters the optimization (for instance, equation (10) has a_{1i} replaced by $a_{1i} + \lambda\alpha_i$), a computational algorithm suggests itself: Starting with λ equal to zero, slowly increase its value, recomputing the optimal z_i^* until constraint (19) is just satisfied.

In the event that the individual risk functions are not convex, or one has two or more side constraints, then the above analysis no longer applies. Usually, however, with a specific problem, the correct constrained solution can be inferred with only a little numerical checking.

THE EFFECT OF INFREQUENT EVENTS

In some applications, it is desirable to investigate the effect of variations in the distribution of the random variable. For instance, if one is planning against an uncertain customer demand, then there may be a possibility of an occasional,

unpredictable surge in demand. How should one change the previously determined decision variable, z^*, to "hedge" against the possibility of this rare event?

Suppose the tail distribution of this occasional happening were given by $Q_1(x)$, and the known probability of its occurrence at every trial were ϵ. If the trials are repeated enough times so that this infrequent event will also occur many times, then the previous analysis applies, with the old distribution replaced by a new, averaged distribution $(1-\epsilon)\,Q(x)+\epsilon Q_1(x)$. One could use the developed formulae to find the exact value of the new optimal decision variable, z^{**}.

However, if the probability ϵ is small, the change in optimal policy will also be small, and one may approximate the new best value of the decision variable by:

$$z^{**} \approx z^* + \epsilon y. \tag{22}$$

We distinguish two cases:

Case I. The values of x associated with the rare event are larger than the normal values of the random variable so that:

$$Q_1(z^{**}) \approx Q_1(z^*) \approx 1$$

and we use the special notation

$$y \overset{\text{def}}{=} y_+. \tag{23}$$

Case II. The variation in the distribution is confined to values of x smaller than the usual values of the random variable, so that

$$Q_1(z^{**}) \approx Q_1(z^*) \approx 0$$

and we define:

$$y \overset{\text{def}}{=} y_-. \tag{24}$$

We see that y_+ would correspond to the case of an occasional increase in customer demand, while y_- would correspond to the possibility of a decrease.

On inserting the averaged distribution into (6), and using the approximations (22) and (23) or (24), we find that in the case of a continuously distributed random variable

$$y_+ = [(m_2 - m_1) - A'(z^*) - D'(z^*)]/R''(z^*), \tag{25}$$

and

$$y_- = -A'(z^*)/R''(z^*). \tag{26}$$

We see that the more sensitive the original solution, the more conservative will be the shift in the value of the decision variable.

For the case in which the risk function is continuous in x, we have the graphical construction shown in Figure 3. After finding the intercepted lengths W_1, W_2, W_3, and W_4, one has simply:

$$y_+ = (W_1 W_2)/(W_1 + W_2) \tag{27}$$

and

$$y_- = -(W_3 W_4)/(W_3 + W_4). \tag{28}$$

In this case, one can also show that the sensitivity of the original solution, $R''(z^*)$, is just the difference of the slopes of the straight lines in Figure 3, i.e. equals $s_1 - s_2$.

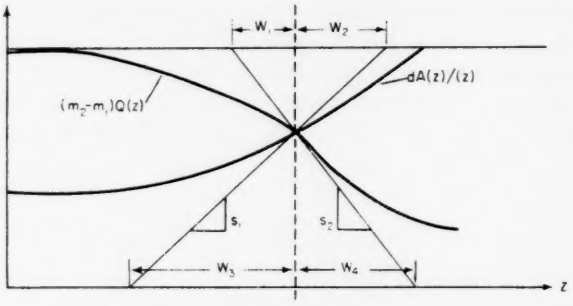

FIG. 3. *Graphical solution for changes in optimal solution.*

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Probably the most interesting characteristic of the solutions (25) and (26) is that they are independent of the actual values of the random variable associated with the rare event, if they are sufficiently far away so that the approximations (23) or (24) hold. Loosely speaking, this means that a change in the optimal solution depends only on the chance that an out-of-the-ordinary event will occur, and not upon its size. For very large shifts in the distribution, of course, the simple linear model is not appropriate.

However, the change in expected risk is not independent of the size of the rare event. For the approximations

$$R(z^{**}) \approx R(z^*) + \epsilon R_+ \quad \text{and} \quad R(z^{**}) \approx R(z^*) + \epsilon R_- \tag{29}$$

which hold for approximations (23) and (24) respectively, we find that the relative shift in minimal risk is given by:

$$R_+ = m_2(\bar{x}_1 - \bar{x}) + D(z^*)\,[1 - Q(z^*)] - (m_2 - m_1)\int_0^{z^*} Q(x)\,dx \tag{30}$$

or

$$R_- = m_1(\bar{x}_1 - \bar{x}) - D(z^*)\,Q(z^*) + (m_2 - m_1)\int_{z^*}^{\infty} Q(x)\,dx \tag{31}$$

where $\bar{x}_1$ is the mean of the distribution $Q_1(x)$ and $\bar{x}$ is the original mean.

In risk problems in which the random variable is distributed discretely, the decision variable will increase only in discrete steps, as the value of ϵ changes. However, the points at which the optimal policy changes can be easily calculated from inequalities (14) or (15). Details are left to the reader.

SOLUTION OF EXAMPLES

We illustrate the application of the techniques just described by solving the problems posed in the Introduction.

I.

In the newsboy problem, let z be the number of papers ordered from the distributor, and x be the customer demand, with known distribution $Q(x)$. If C is the per-unit cost of the newspaper, and S is the per-unit selling price, then the cost function is just:

$$R(z; x) = Cz - Sx \quad (0 \leqslant x < z)$$
$$= (C-S)z \quad (z \leqslant x < \infty).$$

Then:

$$A(z) = Cz \qquad D(z) = 0$$
$$A'(z) = C \qquad m_1 = -S$$
$$A''(z) = 0 \qquad m_2 = 0.$$

From (15), the optimal z^* is found from the conditions:

$$Q(z^*) > C/S$$
$$Q(z^*+1) \leqslant C/S,$$

i.e. the stock-out probability is made as close to $(C/S)+0$ as possible.

For example, suppose that the demand had a Poisson distribution, with a mean of 40 newspapers. If $C/S = 7¢/10¢ = 0{\cdot}7$, then from a table of the tail of the Poisson,† we find:

$$Q(37) = 0{\cdot}703654$$
$$Q(38) = 0{\cdot}645349.$$

The newsboy orders 37 papers each day, and has an expected loss of

$$7{\cdot}37 - 10 \sum_{x=1}^{37} Q(x) = -98{\cdot}317,$$

i.e. his expected profit on a day's operation is about 98¢. This is, of course, less than $(10-7)37 = \$1{\cdot}11$, since on some days he will have a surplus. From a consideration of the optimality conditions, we find that for a possibility of 0·53 per cent of a very small demand, the newsboy will decrease his order, but that it will take a positive rumour of probability 15·65 per cent to make it worth while ordering more papers.

If the newsboy operated another stand with mean demand of 60 papers a day, we find that the optimal order is 56 papers, with an expected profit of \$1·54. Thus, for both news-stands, he orders a total of 93 papers, and has an average total profit of \$2·52.

† *Poisson's Exponential Binomial Limit*, by E. C. Molina. D. Van Nostrand and Company Inc., Princeton, 1942.

Now suppose that his distributor limited him to a total of 80 papers only.[2] By adding this constraint to the two-dimensional problem, we find that the Lagrange multiplier is added directly to the cost C for each sub-problem. If the cost–revenue ratio is the same for each news-stand, then we must find the intersection of the two tail curves (one with mean 40, the other with 60) with the value $0{\cdot}7+\lambda$. We find $\lambda = 0{\cdot}214479, z_1 = 31, z_2 = 49$ and the net profits become 91¢. and \$1·45, respectively. The total profits are reduced to \$2·36 because of the limitations on the total number of newspapers he may order.

We note, incidentally, that if the two news-stands could be combined into one with a (Poisson) demand of mean $= 40+60 = 100$, then an order of 80 papers would bring an average profit of \$2·39. This occurs because there are possible compensations in the fluctuations of the two demand distributions, if they are independent. If they are not independent, then the total demand may not be Poisson.

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II.

Let V be the value of the ore deposit. Then we may write the cost of exploration as

$$V(z/z_0)^2$$

where z is the radius of exploration and z_0 is the "break-even" radius—the radius at which exploration costs just equal the value of the deposit. Letting x be the actual radius at which the ore occurs, and $Q(x)$ the continuous distribution of this distance,

$$\begin{aligned} R(z;x) &= V[(z/z_0)^2-1] \quad (0\leqslant x\leqslant z)\dagger \\ &= V(z/z_0)^2 \qquad\quad (z<x<\infty). \\ A(z) &= V[(z/z_0)^2-1] \qquad D(z) = V \\ A'(z) &= 2Vz/(z_0)^2 \qquad D'(z) = 0 \\ A''(z) &= 2V/(z_0)^2 \qquad m_1 = m_2 = 0. \end{aligned}$$

The optimal policy is to explore to a radius z^* such that:

$$Vp(z^*) = 2Vz^*/(z_0)^2$$

with a sufficiency condition for a minimum of

$$V\frac{dp'(z^*)}{dz} < 2V(z_0)^2.$$

† The risk function for this problem has a behavior different for $x = z$ than that assumed in the development of the method. For continuously distributed variables, this does not matter; for situations in which x has discrete components, one may easily make the necessary modifications.

A graphical solution is shown below in Figure 4. The expected risk is just the difference in areas between the two curves up to z^*, i.e. $R(z^*) =$ Area I $-$ Area II.

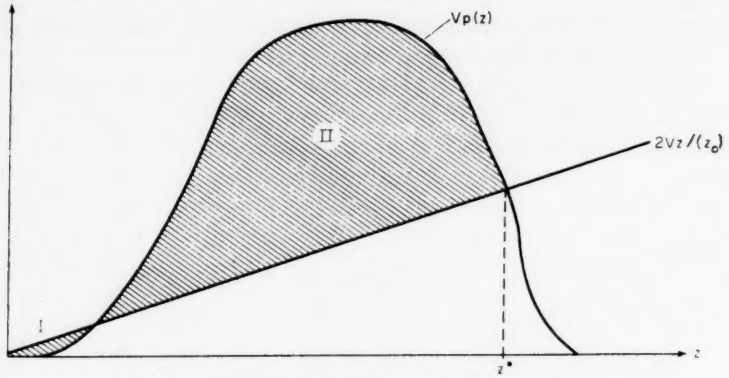

FIG. 4. *Graphical solution for exploration problem.*

III.

Let the range of bar stock length of interest be (0, 1), and let the demand for stock of length x be given by the distribution function $Q(x)$. If C is the cost per unit length of the finished product, and S is the salvage value per unit, we see that the unit cost (as normalized by the given range of lengths) is:

$$CL - S(L - \bar{x}).$$

If z is the new length of bar to be carried in stock, then the per-unit risk (cost) function is just:

$$\begin{aligned} R(z; x) &= (C-S)z + Sx \quad (0 \leqslant x \leqslant z) \\ &= (C-S)L + Sx \quad (z < x < \infty). \end{aligned}$$

(See the footnote to Problem II.)

Thus: $A(z) = (C-S)z$ $\quad D(z) = (C-S)(L-z)$ $\quad m_1 = m_2 = S$

$A'(z) = (C-S)$ $\quad D'(z) = -(C-S)$

$A''(z) = 0$ $\quad D''(z) = 0.$

The optimal length of intermediate-size stock, z^*, is given by the solution of:

$$(C-S)[1 - Q(z^*)] = (C-S)(L-z^*)p(z^*)$$

with a sufficiency condition for a minimum of

$$(C-S)[2p(z^*) - (L-z^*)p'(z^*)] > 0.$$

We note that optimal value of z is independent of the cost of stock and its salvage value! It is also independent of any profit charged according to the length x.

A convenient graphical construction is shown in Figure 5. A point $z = z^*$ is found on the graph of $(C-S)\,Q(z)$ such that the two distances w_1 and w_2 are equal.

If such a point exists, then there is always a decrease in the expected cost per order which is given by the shaded area in Figure 5.

A curious result is that while $y_- = -(C-S)/R''(z^*)$, y_+ is identically zero. We may interpret this by saying that a small shift in demand of bar stock towards larger lengths induces no first-order changes in z^*.

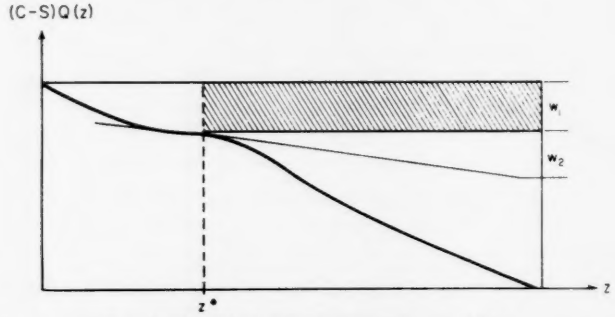

FIG. 5. *Graphical solution to stockage problem.*

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A Short Cut in a Class of Simulation Problems

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This paper presents a short cut for simulation of a queueing situation in which customers arrive in large batches, more particularly for the case where there are several servers and it is material that these will not come to the end of their work simultaneously.

IN many industrial queueing problems, "customers" consist of jobs that arrive in batches at intervals. A "server" is some sort of operative or machine, which works them off; and the chief point of interest is the extent to which work is carried over from one delivery to the next or alternatively the amount of idle time spent by the servers through running out of work before a new delivery arrives. The number of jobs required is determined by some process, random or otherwise, not relevant here. The lengths of these jobs are to be taken as random drawings from a known distribution. If the number of jobs per delivery is at all large, the drawing of the job lengths and the simulation of the way they are worked off can become very time consuming. This paper presents a short cut to deal with such cases.

For example, in a problem recently handled by the author, a work shifter brought batches of work to a group of inspectors, the intervals between deliveries and the number of jobs brought both being random. For any delivery, the interval since the previous one and the number of jobs brought were each determined by a random number. The number of jobs per delivery was of the order of 30 to 60, which would have made the drawing of the lengths of the individual jobs and the simulation of their consumption of the inspectors' time a long and tedious process. The following short cut was devised to reduce the work.

In the case of one server, the problem is simplicity itself. Let us suppose that the distribution of job lengths from which we are drawing has frequency function $f(l)$, mean $\bar{l}$ and standard deviation σ. Then if we wish to find the equivalent of the sum of n values of l drawn at random, for reasonably large values of n we can take a single drawing from the normal distribution to which the sum tends. In fact, we draw one figure r from a table of random normal deviates (instead of n random numbers) and calculate $n\bar{l}+r.\sqrt{(n)}\,\sigma$. (If the original distribution has unduly high third or fourth moments, it is not difficult to correct for these by assuming the sum to be better approximated to by a Type A distribution;† but this refinement is unlikely to be worth introducing.) Then,

† A Type A distribution is one the distribution function of which has the form $\Phi(x;\mu,\sigma)+\alpha\Phi^{III}(x;\mu,\sigma)+\beta\Phi^{IV}(x;\mu,\sigma)$ where $\Phi(x;\mu,\sigma)$ is the distribution function for the normal distribution with mean μ and standard deviation σ.

as we have only one server, he is taken to work off the whole batch as if one were simulating one single large job. If he finishes before the new delivery, the idle time is noted; and if he still has work in hand at the new delivery, this is noted as carry over and worked off before the new batch is started. There are no complications here.

If, however, there are j servers, where j is greater than 1, we may assume that as each server finishes one job he takes the next one from those waiting, as long as there are any. It may be that a new delivery will arrive while all servers are still at work. On the other hand, one or more of the servers may come to the end of a job and find no more work waiting; and at times some may do so while the others are engaged on jobs long enough to carry them past the next delivery. If all servers are still at work when a new delivery arrives the simulation may be carried on by simply adding to the aggregate time needed to work off the new batch the length of all work remaining at the delivery time. If, on the other hand, all servers run out of work before a fresh delivery, then the idle time is simply the difference between the total working hours of all servers between the last and the current delivery and the length of work before the servers after delivery of the last batch. What we require, therefore, is a means of estimating whether all servers will still be at work when a new delivery arrives; and if not, the lengths of time that will be carried over by any servers after the arrival of the next batch.

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Let us suppose that the total of work in hand (new delivery plus carry-over) immediately after a delivery is T. Further, let the time that elapses from this delivery till the first instant at which a server completes a job and finds none of this lot of work to go on with be t_0. (It is possible that by this time a new delivery will have arrived: it is assumed, however, that jobs are taken on a first in—first out basis.) Further, let the times taken by the other servers to complete the jobs they are occupied with after the period be $t_1, \ldots, t_{j-1}$. Thus we have that

$$T = jt_0 + \sum_{i=1}^{j-1} t_i.$$

If, moreover, the interval between this delivery and the next is d, if t_0 exceeds d, then no server has any idle time, and the amount of work carried over is $T-jd$. If, however, t_0 is less than d, then the amount of work carried over is $\Sigma(t_0+t_i-d)$, the sum being taken over all i for which this difference is positive, while the amount of idle server time is $d-t_0+\Sigma(d-t_0-t_i)$, the sum being taken over all other values of i.

The essential step in the short cut depends on the fact that it is possible to draw values for t_i from a distribution derived from the known distribution of job lengths (hereinafter to be called the Derived Distribution). Since we are assumed to know T (determined as in the paragraph above), t_0 can be found from

the equation

$$T = jt_0 + \sum_{i=1}^{j-1} t_i,$$

and the amounts of idle time and work carried over follow as in the last paragraph.

Provided that the number of jobs handled by each server is moderately large and there are no startling regularities in the distribution of job times, the times at which the various servers started on their final jobs will bear no relation to one another: they will have been effectively randomized. The last job of each server in each batch is of course a random drawing from the distribution of all jobs. Hence t_i consists of such a random drawing less the amount of time already spent before the first server to finish becomes idle. The instant when this occurs, however, may be anywhere during the job from start to finish, with equal probability. (It may make this idea clearer to point out that if the distribution of job lengths is, instead, taken as being the distribution of intervals between arrivals of buses at a stop, the required distribution is that of the times intending passengers must wait, if they arrive at the stop without knowledge of the vagaries of the service.)

In the appendix we show that, if the distribution of job lengths has frequency function $f(l)$ and mean $\bar{l}$, the frequency function for the Derived Distribution is

$$\frac{1}{\bar{l}} \int_t^\infty f(l)\, dl.$$

Even if no analytical form can be found for $f(l)$, it is easy to draw up the usual sort of table by which appropriate random numbers are assigned to particular values of t. From the given table of frequencies of job lengths, we make another, showing the number of jobs observed with lengths greater than each given value. (The figures for job lengths must be equally spaced.) Then the entries in this table may be taken as a random sample from the required distribution, and converted to percentages by dividing by the sum of all the values in the table.

The following example may make this clearer. In the Table, column 1 gives the observed lengths of job, and column 2 the number of jobs of these lengths observed. Column 3 is the sum of the values in column 2 up to but not including the same row, and column 4 the same figures as percentages of their sum. Then in the usual way we assign random numbers 01 to 18 to an interval of 0, 19 to 35 to an interval of 1, and so on, the number 00 corresponding to a carry over of 21·9.

Job length	Frequency	Cumulated values from (2)	(3) as % of total	Random numbers assigned	Carry-over time (t_i)
0	—	231	18·1	01–18	0
1	16	215	16·9	19–35	1
2	30	185	14·5	36–50	2
3	45	140	11·0	51–61	3
4	33	107	8·4	62–69	4
5	20	87	6·8	70–76	5
6	17	70	5·5	77–81	6
7	12	58	4·6	82–86	7
8	15	43	3·4	87–89	8
9	10	33	2·6	90–92	9
10	9	24	1·9	93–94	10
11	5	19	2·5	95–96	11·4
12	6	13			
13	2	11	2·0	97–98	14·4
14	3	8			
15	1	7			
16	2	5	0·9	99	16·8
17	1	4			
18	1	3			
19	0	3			
20	1	2			
21	0	2	0·9	00	21·9
22	0	2			
23	1	1			
24	0	1			
25	1	—			
TOTAL	231	1274			

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APPENDIX

Author's note: The results here are probably well known, but I have seen no published reference to them.

Let us suppose we have a large number N of intervals. The aggregate of the lengths of all these intervals will be $N\bar{l}$, and the number of intervals of length between l and $l+\delta l$ will be $Nf(l)\,\delta l$.

Of this total length $N\bar{l}$, the amount that precedes the next-following end of an interval by a period of between t and $t+\delta t$ will be δt in each interval of length $t+\delta t$ or more, but zero in all shorter intervals; thus, over the whole

distribution, it will be

$$N\delta t\int_{t+\delta t}^{\infty} f(l)\,dl.$$

As a fraction of the total length $N\bar{l}$, this becomes

$$\frac{\delta t}{\bar{l}}\int_{t+\delta t}^{\infty} f(l)\,dl.$$

Proceeding to the limit, we see that the frequency function for t is

$$\frac{1}{\bar{l}}\int_{t}^{\infty} f(l)\,dl.$$

It is worth noting that if we know the moments of the distribution of those, the distribution of t may readily be found. For the characteristic function of the latter is

$$\int_{\theta}^{\infty} e^{ixt}\,dt\int_{t}^{\infty}\frac{f(l)\,dl}{\bar{l}}.$$

For any frequency function $f(l)$ this can be integrated by parts to give

$$\left[\frac{1}{i\bar{l}x}e^{ixt}\int_{t}^{\infty} f(l)\,dl\right]_{0}^{\infty}+\frac{1}{i\bar{l}x}\int_{0}^{\infty} l^{ixt}f(t)\,dt,$$

i.e.
$$\frac{1}{i\bar{l}x}\{f^*(x)-1\},$$

where $f^*(x)$ is the characteristic function corresponding to $f(l)$.

In particular, if the distribution of l has mean $\bar{l}$ and second and third moments about the mean σ^2, μ_3,

$$f^*(x)=e^{i\bar{l}x}\left(1-\frac{\sigma^2x^2}{2!}-\frac{i\mu_3x^3}{3!}\ldots\right)$$

$$=1+i\bar{l}x-\frac{\bar{l}^2+\sigma^2}{2!}x^2-i\frac{\mu_3+3\bar{l}\sigma^2+\bar{l}^3}{3!}x^3\ldots.$$

Hence
$$\frac{f^*(x)-1}{i\bar{l}x}=1+i\left(\frac{\bar{l}}{2}+\frac{\sigma^2}{2\bar{l}}\right)x-\frac{\mu_3+3\bar{l}\sigma^2+\bar{l}^3}{3!\,\bar{l}}x^2\ldots,$$

from which we see that the mean value of t is $(\bar{l}/2)+(\sigma^2/2\bar{l})$, and the variance of t is

$$\frac{\sigma^2}{2}+\frac{\bar{l}^2}{12}+\frac{\mu_3}{3\bar{l}}-\frac{\sigma^4}{4\bar{l}^2}.$$

Scheduling for Known but Irregular Batch-Wise Demand

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The paper deals with the batch size problem, given that demand is batch-wise and the times and quantities of the first n demands are known. This is found to be a problem in dynamic programming.

A treatment is devised which may often enable that part of the demand to be scheduled optimally which must be procured before further information becomes available; which solves the problem in such cases. In other cases the final decision between the surviving possibilities may have to be based on possibly non-optimal criteria. Where it is known that the nth demand is the last, the problem can be solved completely.

Details of computational methods are given, including worked examples. The methods appear suitable for computer programming, but are in any case (relatively) fast by hand; a solution may normally be obtained in, say, $5n$ minutes or less, except where the individual demands are much smaller than the optimum batch size (in which case other—approximate—methods may be preferred and would be unlikely to incur much penalty).

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INTRODUCTION

THE paper discusses a variant of the general Scheduling or Economic Lot Size problem in which one has the object of finding that order or batch size which will yield the lowest total of set-up and stock-holding costs. In most applications the demand is, of course, stochastic; in the remainder where the demand is known it can often be taken as continuous at a steady rate with negligible penalty, and hence standard batch size formulae can be applied. Nevertheless, there may be cases where the known demand is distributed so irregularly in time that a calculation of the actual optimum schedule becomes worth while. This is the problem discussed in the paper, with the particular object of devising quick and simple methods suitable for routine application. (The point here is that highly irregular demand patterns are often not repetitive, so that a calculation may have to be carried out at frequent intervals.)

FORMULATION OF PROBLEM

We suppose that information about future demand takes the form of a programme,

$$(q_1, T_1), \quad (q_2, T_2), \quad \ldots, \quad (q_n, T_n),$$

where q_i is the quantity required at time T_i. To satisfy this demand it is clearly optimal to use up any existing stocks first, and we assume that the programme already excludes any demand which can be so satisfied.

Of the various possible schedules which can be devised to meet this programme we need only consider those which take the form of successive batches or

orders of quantities

$$(q_1+\ldots+q_{r-1}),\quad (q_r+\ldots+q_{s-1}),\quad (q_s+\ldots+q_{t-1}),\ldots,$$

to be available at times $T_1, T_r, T_s, \ldots$, respectively; in particular it cannot be optimal to have a batch quantity equal to $(q_i+\ldots+q_{j-1}+p)$ where $p<q_j$, for it would then be possible to save stock-holding costs without increasing the number of batches. Hence scheduled batches or orders must include either the whole of any q_j, or none of it. Similarly it cannot be optimal to have any of the scheduled batches or orders ready before they are required.

In deciding between different possible schedules, we assume as usual that the only factors affected by the decision are: a cost element proportional to the number of batches made (the "set-up cost"), and a cost element proportional to the product of time and quantity of stock held (the "stock-holding cost"). It is then well established that the optimum schedule is that which has the lowest total of set-up and stock-holding costs, provided the latter is correctly defined to include the critical marginal rate of return on capital invested in stock as well as any out-of-pocket costs of carrying stock such as warehouse space.

It is useful at this stage to distinguish between the "closed" case and the "open" case. In the "closed" case, the given programme $(q_1, T_1), \ldots, (q_n, T_n)$ comprises the whole of future demand (or, equivalently, it is known that no further demand will arise after T_n, at least not within any period for which it might conceivably be optimal to carry stock). In that case the total of the scheduled batches or orders must equal the total of requirements $q_1+q_2+\ldots+q_n$. There is one way of making this in one batch, $n-1$ ways of making it in 2 batches, ...,

$$\frac{(n-1)!}{(n-r)!(r-1)!}$$

ways of making it in r batches, ..., i.e. there are altogether 2^{n-1} possible schedules. Clearly the relevant costs can be determined for each and one or more optimum schedules exist and can be found. The treatment below aims at a rapid and systematic way of arriving at the optimum schedule (or schedules) for the "closed" case.

In the "open" case, demand is expected to continue after T_n and the programme in effect provides only part of the total information about future demand. In that case it cannot be assumed that the total of scheduled batches or orders should equal the total of the given q's and the problem is not fully defined. The "open" case is, of course, the one often encountered in practice, where a demand programme may be given for a certain period ahead, to be followed later by a new programme for the next period, and so on.

In such cases the methods developed below for the closed case can still be used and will normally eliminate all but a relatively small number of potentially

optimal schedules. The further treatment of the open case after this stage has been reached is discussed in more detail below, but it may be noted that a schedule which is optimal irrespective of the form taken by demand after T_n will exist only in special cases and that a possibly non-optimal criterion (such as lowest cost per piece scheduled) may have to be used.

SOLUTION—CLOSED CASE

1. *The Z-Method*

Let C = set-up cost for 1 batch.

d = unit stock-holding cost (unit quantity for unit time). (For convenience express cost in terms of d; the set-up cost is then C/d and the unit stock-holding cost is 1.)

q_i = amount of ith demand in given demand programme.

$Q_i = q_1 + \ldots + q_i$; $Q_{ij} = Q_i - Q_{j-1} = q_j + \ldots + q_i \quad (0 < j \leqslant i)$.

T_i = time at which ith demand is required.

$t_i = T_i - T_{i-1}$.

Z_{ij} = lowest possible cost (set-up and stock-holding) of producing the first Q_i in the demand programme by a schedule in which the last batch is Q_{ij}.

M_i = lowest possible cost (set-up and stock holding) of producing the first Q_i in the demand programme.

From the definitions, $$M_i = \operatorname{Min} Z_{ij} \quad (0 < j \leqslant i), \tag{1}$$

Z_{ij} is the cost of that schedule for the first Q_i which

(a) has Q_{ij} as its last batch

(b) has the lowest cost of all such schedules.

Hence Z_{ij} is M_{j-1} plus the set-up and stock-holding cost of the batch Q_{ij}. This batch must have been produced at time T_j and hence its stock-holding cost was

$$t_{j+1}Q_{i,j+1} + t_{j+2}Q_{i,j+2} + \ldots + t_iQ_{i,i},$$

i.e. $$\sum_{j+1}^{i} t_rQ_{i,r} = \sum_{j+1}^{i} q_r(T_r - T_j)$$

and its set-up cost was C/d.

Hence $$Z_{ij} = \sum_{j+1}^{i} t_rQ_{i,r} + C/d + M_{j-1} \quad (0 < j \leqslant i). \tag{2}$$

Evidently a table of Z_{ij} will solve the problem; for if all the Z_{nj} are known, the smallest of them (say, $Z_{nk} = M_n$) can be found and hence k can be determined; this means that the last batch in the required optimum schedule is

$$Q_{nk} = q_k + \ldots + q_n.$$

The preceding batch can then be determined similarly by identifying the smallest of the $Z_{k-1,j}$, and so on, until the whole schedule has been constructed.

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From (2) we have:

For $0<j<i$, $Z_{ij} = Z_{i-1,j}+q_i(T_i-T_j)$. (2a)

For $0<j=i$, $Z_{ij} = Z_{ii} = M_{i-1}+C/d$. (2b)

Therefore to compute a Table of $Z_{ij}(0<j\leqslant i)$, the entry in the ith row, jth column is obtained (for $0<j<i$) by adding $q_i(T_i-T_j)$ to the entry in the square above, and for $i=j$ (i.e. those along the diagonal of the Table) by adding C/d to the smallest entry in the preceding row. The process of computation starts with C/d entered in the 1st row, 1st column.

Next, let r, s be given by $Z_{i-1,r} = \operatorname{Min} Z_{i-1,j} = M_{i-1}$

and $Z_{i,s} = \operatorname{Min} Z_{i,j} = M_i$.

Since $Z_{i-1,r}\leqslant Z_{i-1,s}$,

$$Z_{i-1,r}+q_i(T_i-T_r)\leqslant Z_{i-1,s}+q_i(T_i-T_s)+q_i(T_s-T_r),$$

i.e. $Z_{i,r}\leqslant Z_{i,s}+q_i(T_s-T_r)$;

but $M_i = Z_{i,s}\leqslant Z_{i,r}$;

hence $q_i(T_s-T_r)\geqslant 0$ and $s\geqslant r$. (2c)

This means that once the column containing the smallest Z in a given row has been found, no Z's in subsequent rows to the left of that column need be computed, for they could not be row minima. The volume of calculation may sometimes be reduced by making use of this result.

TABLE 1. Z-TABLE

i	q_i	T_i						
1	10	0	(200)					
2	5	3	(215)	400				
3	20	8	(375)	500	415			
4	3	10	(405)	521	421	575		
5	6	11	471	569	(439)	581	605	
6	30	12	—	—	(559)	641	635	639
		j	1	2	3	4	5	6
		T_j	0	3	8	10	11	12

Table 1 shows the working for a simple demand programme of 6 demands, (10,0), (5,3), (20,8), (3,10), (6,11), (30,12).

C/d has been taken as 200. The row minima are shown in brackets.

Since $\text{Min}\, Z_{nj} = \text{Min}\, Z_{6,j} = Z_{6,3}$, the last batch in optimum schedule is $q_3+q_4+q_5+q_6$. $\text{Min}\, Z_{2,j} = Z_{2,1}$, so that the preceding batch is q_1+q_2. The optimum schedule is therefore: make 15 at time $T_1 = 0$ and 59 at time $T_3 = 8$.

From a practical point of view this method is by no means laborious (considering the nature of the problem) but was felt to be capable of improvement. The main labour involved is the calculation of the $q_i(T_i - T_j)$ to be added to the $Z_{i-1,j}$ by (2a); also, the Z's in the lower rows tend to become inconveniently large, though this effect is not serious in the simple case above with only 6 demands. However, since the only object of the Z calculation is to identify the row minima, it seemed desirable to investigate whether this could be done with less labour.

2. *The D-Method*

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Let
$$D_{ij} = Z_{ij} - Z_{ij-1} \tag{3}$$
and consider Table 2. This is a D-Table for the same demand programme, obtained from Table 1 by taking the differences between adjacent Z's. Brackets are shown in the squares which correspond to the row minima in Table 1.

TABLE 2. *D*-TABLE

$j =$	1	2	3	4	5	6
$i = 1$	( )					
2	( )	185				
3	( )	125	−85			
4	( )	116	−100	154		
5		98	(−130)	142	24	
6		—	(—)	82	−6	4

First consider how the row minima in Table 1 could have been identified in Table 2 without reference to Table 1.

As before let s be the column containing the smallest Z in the ith row.

In Table 1 where the Z's are available the process of finding s is probably too obvious to require discussion. If, however, one analyses this process, one possible method is seen to be the following:

Proceeding from right to left, compare each Z in the ith row in turn with $Z_{i,p}$, the latter being the smallest Z already scrutinized. A given $Z_{i,j}$ will either be exceeded by $Z_{i,p}$ or it will not. If it is not, retain the same $Z_{i,p}$ and continue the process with the next Z on the left (i.e. $Z_{i,j-1}$). If it is, j becomes p and the

process is continued with the next Z on the left. The first p is i and the last surviving p (after $j = 1$) is s.

Now $$Z_{i,p} - Z_{i,j} = \sum_{j+1}^{p} D_{i,x} \quad (\text{for } p > j);$$

therefore $$Z_{i,p} > Z_{i,j} \quad \text{means that} \quad \sum_{j+1}^{p} D_{i,x} > 0,$$

and $$Z_{i,p} \leqslant Z_{i,j} \quad \text{means that} \quad \sum_{j+1}^{p} D_{i,x} \leqslant 0.$$

The above process is therefore equivalent to the following:

Proceeding from right to left in the ith row of D's add them cumulatively to obtain in succession

$$D_{i,p},\ D_{i,p} + D_{i,p-1},\ D_{i,p} + D_{i,p-1} + D_{i,p-2},\ \ldots,$$

and so on.

After each addition (say, that of $D_{i,j}$) the cumulative sum will either be positive or it will not. If it is not, continue the process by adding the next D on the left (i.e. $D_{i,j-1}$). If it is, p becomes $j-1$ and the process is restarted (i.e. with $D_{i,j-1},\ D_{i,j-1} + D_{i,j-2},\ \ldots$, and so on). The first p is i and the last surviving p (after $j = 2$) is s.

(In view of (2c) it is evidently not necessary to continue after $j = r+1$, where $Z_{i-1,r} = M_{i-1}$.)

Inspection of Table 2 shows that this process would in fact have led to the bracketed square in each row.

In symbols the argument is:

$$\begin{aligned} s &= p \quad (p = s) \\ &= p(Z_{i,p} = \min_{0<j\leqslant i} Z_{i,j}) \\ &= p(Z_{i,p} \not> \min_{0<j\leqslant i} Z_{i,j}); \end{aligned}$$

but if $$Z_{i,p} \not> \min_{0<j\leqslant p} Z_{ij},$$

then either $$Z_{i,p} \not> \min_{0<j\leqslant p} Z_{i,j} \quad \text{or} \quad s > p;$$

therefore $$\begin{aligned} s &= \max_{0<p\leqslant i} p(Z_{i,p} \not> \min_{0<j\leqslant p} Z_{i,j}) \\ &= \max_{0<p\leqslant i} p(Z_{i,p} - \min_{0<j\leqslant p} Z_{i,j} \not> 0) \\ &= \max_{0<p\leqslant i} p\left(\max_{0<j<p} \sum_{j+1}^{p} D_{i,x} \not> 0\right), \end{aligned}$$

and in view of (2c) $$s = \max_{r\leqslant p\leqslant i} p\left(\max_{r\leqslant j<p} \sum_{j+1}^{p} D_{i,x} \not> 0\right). \qquad (3a)$$

For checking purposes it may be useful to have an independent method of determining s. This can be done by basing the process on a scrutiny of the Z's in the ith row from left to right (instead of from right to left). Analysing this as before leads to the conclusion that the D's have to be added cumulatively from left to right to form in succession the terms

$$D_{i,p+1},\ D_{i,p+1}+D_{i,p+2},\ \ldots,\quad \text{and so on;}$$

the process continues for as long as the cumulative sum is not negative; if (after, say, $D_{i,j}$ has been added) it becomes negative, j becomes p and the addition is restarted (with $D_{i,j+1}$, $D_{i,j+1}+D_{i,j+2}$, ..., and so on); the first p is 1 and the last surviving p (after adding $D_{i,i}$) is s. In this form the process is represented by

$$s = \min_{r\leqslant p\leqslant i} p\left(\min_{p<j\leqslant i} \sum_{p+1}^{i} D_{i,x} \not< 0\right). \tag{3b}$$

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In practice it is found that whether the relevant cumulative sums of D's (which often involve only one or two terms) are positive or negative can be determined by inspection without lengthy calculation, and after some little practice the determination of the row minima in a D-Table is almost as fast as in a Z-Table.

It remains to consider how the D's in Table 2 could have been calculated without reference to Table 1, and in particular whether the computation of a D-Table shows advantages compared with that of a Z-Table.

For $1<j<i$,
$$\begin{aligned} D_{i,j} &= Z_{i,j}-Z_{i,j-1} \\ &= Z_{i-1,j}+q_i(T_i-T_j)-Z_{i-1,j-1}-q_i(T_i-T_{j-1}) \\ &= Z_{i-1,j}-Z_{i-1,j-1}-q_i(T_j-T_{j-1}) \\ &= D_{i-1,j}-q_i t_j. \end{aligned} \tag{4a}$$

Again, let r, s be given by $Z_{i-1,r}=M_{i-1}$ and $Z_{i,s}=M_i$, as before: then, for $j=i$,

$$\begin{aligned} D_{i,j} = D_{i,i} &= Z_{i,i}-Z_{i,i-1} \\ &= M_{i-1}+C/d-Z_{i-1,i-1}-q_i(T_i-T_{i-1}) \\ &= C/d-\sum_{r+1}^{i-1} D_{i-1,j}-q_i t_i. \end{aligned} \tag{4b}$$

Hence each D can be computed by subtracting the relevant $q_i t_j$ from the D already calculated in the square above it (for $j<i$), or from C/d less the sum of the D's to the right of the rth column in the row above (for $j=i$). The computation for $Z_{i,i}(=M_{i-1}+C/d)$ is easier than for $D_{i,i}$, but the difference is not great and applies only to one term in each row; on the whole this tends to be outweighed by the greater ease of computing $q_i t_j$ for (4a) as compared with $q_i(T_i-T_j)$ for (2a), for all the remaining terms.

The following additional results may be useful:

Let
$$L_i = \sum_{r+1}^{s} D_{i,j}, \tag{4c}$$

$$S_i = q_i(T_i - T_r) = \sum_{r+1}^{i} q_i t_j. \tag{4d}$$

(Both r and s will be known, and the relevant $q_i t_j$ will already have been calculated to obtain the ith row of D's, so that L_i and S_i will be available without much extra labour.)

Then
$$L_i + S_i + \sum_{s+1}^{i} D_{i,j} = S_i + \sum_{r+1}^{i} D_{i,j} = Z_{i,r} - Z_{i-1,r} + Z_{i,i} - Z_{i,r}$$
$$= Z_{i,i} - M_{i-1}$$
$$= C/d. \tag{4e}$$

Also, $L_i + S_i = C/d - \sum_{s+1}^{i} D_{i,j}$, so that (4b) becomes

$$D_{i,i} = S_{i-1} + L_{i-1} - q_i t_i. \tag{4f}$$

Finally,
$$L_i + S_i = C/d - \sum_{s+1}^{i} D_{i,j}$$
$$= Z_{i,i} - M_{i-1} - (Z_{i,i} - Z_{i,s})$$
$$= M_i - M_{i-1}. \tag{4g}$$

It may therefore be convenient to calculate L_i and S_i in any event; $D_{i,i}$ can then be obtained by either (4b) or (4f), (4e) can be used as a check, and the cost of the optimum schedule obtained from (4g), which gives

$$M_n = C/d + \sum_{2}^{n} (L_i + S_i).$$

From a computational point of view, the D-method appears somewhat the better of the two, at least for large n. Some detailed worked examples will now be given.

Worked Examples—D-Method

A somewhat different lay-out from that in Table 2 has been found convenient. This is shown in Table 3. The demand programme containing 11 demands (the first 6 of which are, however, identical with those already used) is shown in the two lower centre rows of the Table. C/d was again taken as 200 and entered in the first space in the top centre row (headed $M_i - M_{i-1} = L_i + S_i$). The S_i are entered in the bottom row and the L_i in the top row. The lower part of the Table is used for the $q_i t_j (r < j \leqslant i)$ and the rows are headed $j = i$,

$j = i-1$, etc. as shown. The corresponding D_{ij} are entered in the upper part of the Table.

To obtain the ith column of D_{ij}, the procedure is:

(1) "Line off" the $D_{i-1,j}$ in the preceding column and obtain r and L_{i-1}.

(2) Add the $q_{i-1}t_j$ in the preceding column and obtain S_{i-1}; also obtain $S_{i-1}+L_{i-1}$.

(3) Calculate the q_it_j for $r<j\leqslant i$.

(4) Obtain the D_{ij} for $r<j\leqslant i$ from the $D_{i-1,j}$ and q_it_j (from (4a) and (4f)). Then go on to the next column.

TABLE 3. DEMAND PROGRAMME (n = 11)

L_i		0	0	0	−32	0	0	0	−136	−44	0
									26		
								56	−34		
D_{ij} $j=i-3$					98		72	−19	−24		
$j=i-2$				116	−130	82	−11	−9	−104	4	
$j=i-1$			125	−100	142	−6	−1	16	14	−48	74
$j=i$		185	−85	154	24	4	80	44	−8	164	−24
$M_i-M_{i-1}=L_i+S_i$	200	15	160	30	34	120	60	112	194	36	150
q_i	10	5	20	3	6	30	5	8	5	5	15
t_i		3	5	2	1	1	8	2	8	6	4
$j=i$		15	100	6	6	30	40	16	120	30	60
$j=i-1$			60	15	12	30	5	64	30	40	90
q_it_j $j=i-2$				9	30	60	5	8	120	10	
$j=i-3$					18		10	8	15		
								16	15		
									30		
S_i		15	160	30	66	120	60	112	330	80	150
$i=$	1	2	3	4	5	6	7	8	9	10	11

1. "Lining off". Draw a line above the top $D_{i-1,j}$ in the preceding column. Also draw a line below each $D_{i-1,j}$ (working from the top of the column down) such that the sum of the $D_{i-1,j}$ between that line and the next line above is negative. All $D_{i-1,j}$ above the lowest such line which can be drawn are termed "lined off"; the lowest $D_{i-1,j}$ which has been "lined off" is $D_{i-1,r}$ where r satisfies (3b) (applied to the $(i-1)$th column). L_{i-1} is the sum of the "lined off" $D_{i-1,j}$ and is entered in the top row. No q_it_j or D_{ij} are calculated in subsequent columns for any j for which $D_{i-1,j}$ was "lined off".

2. S_{i-1} is the sum of the calculated $q_{i-1}t_j$ and is entered in the bottom row, hence obtain $S_{i-1}+L_{i-1}$ and enter this in the top centre row.

3. Calculate q_it_i and enter this in the top row in the lower part of the Table; below this enter q_it_{i-1}, and so on, for one more product than the number of $D_{i-1,j}$ not lined off in the preceding column. (Note: If the given q's and t's are

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entered as shown, each $q_i t_j$ will be the product of the q at the head of the column and the t at the head of the $(i-j)$th column on the left.)

4. Subtract $q_i t_i$ from $L_{i-1}+S_{i-1}$ in the preceding $(i-1)$th column. This is D_{ii} and is entered in the ith column but in the row above L_i+S_i. Similarly subtract the remaining $q_i t_j$ (if any) from the $D_{i-1,j}$ not lined off in the preceding column, each D_{ij} obtained in this way being entered in the space diagonally above the corresponding $D_{i-1,j}$.

Thus, for $i=2$, $q_2 t_2 = 15$, $D_{22} = 200-15 = 185$.

for $i=3$, no D's in column 2 can be lined off, hence $L_2 = 0$; $S_2 = 15$;
$L_2+S_2 = 15$; $q_3 t_3 = 100$, $q_3 t_2 = 60$;
$D_{33} = 15-100 = -85$, $D_{32} = 185-60 = 125$.

for $i=4$, no D's in column 3 can be lined off, hence $L_3 = 0$; $S_3 = 160$;
$L_3+S_3 = 160$; $q_4 t_4 = 6$, $q_4 t_3 = 15$, $q_4 t_2 = 9$;
$D_{44} = 160-6 = 154$, $D_{43} = -85-15 = -100$, $D_{42} = 125-9 = 116$.

for $i=5$, we have, proceeding as before,
$D_{55} = 24$, $D_{54} = 42$, $D_{53} = -130$, $D_{52} = 98$.

for $i=6$, we find that $D_{52}+D_{53} = 98-130 = -32$, so these are lined off. No further D's in column 5 can be lined off, hence $L_5 = -32$; $S_5 = 66$; $L_5+S_5 = 34$; $q_6 t_6 = 30$, $q_6 t_5 = 30$, $q_6 t_4 = 60$, $q_6 t_3$ and $q_6 t_2$ are not calculated since D_{53} and D_{52} were lined off;
$D_{66} = 4$, $D_{65} = -6$, $D_{64} = 82$; and so on.

When the last column of D_{ij} (i.e. the D_{nj}) have been obtained, complete the work by lining it off and entering S_n+L_n.

Finally obtain the optimum schedule. The top D_{ij} not lined off in the last (11th) column is $D_{i,i-1} = D_{11,10}$; hence the last batch is $q_9+q_{10}+q_{11}$. Again, in the 8th column, the top D_{ij} not lined off is $D_{i,i-4} = D_{8,4}$; hence the penultimate batch is $q_3+\ldots+q_8$. Lastly, in the 2nd column the top D_{ij} not lined off is D_{22} and the antepenultimate batch is q_1+q_2. This is the first batch in the schedule.

The (set-up and stock-holding) cost of this schedule is

$$d(200+15+\ldots+150) = 1111d.$$

Checking

A number of arithmetical checks are possible if the calculations are carried out as described above. Some of these are obvious, but the following may be mentioned.

The $q_i t_j$ may be checked by dividing S_i by q_i for each column. (In those columns where any D_{ij} were lined off, obtain also the sum of the corresponding $q_i t_j$, divide this by q_i and add the quotient to S_r/q_r for all subsequent columns.) These must equal the given T_i, or their successive differences the given t_i.

The D_{ij} may be checked by adding all the D_{ij} and all the $q_i t_j$ in a column. The total must be C/d, for each column (cf. Eq. (4e)).

Again, the sum of the row totals for L_i and S_i must equal the row total $M_i - M_{i-1}$ less C/d.

"Lining off" may be checked as follows: Going up each column of D_{ij}, tick each D_{ij} if the sum of that D_{ij} and those (if any) not ticked below it, but not below the next ticked D_{ij} below it, is positive. There must be no ticked D_{ij} which has been lined off, and the top D_{ij} not lined off in each column must be ticked.

Shortened Working

If it is felt that detailed checking will not be required, the working may be shortened by omitting the lower part of the Table entirely; the $q_i t_j$ are, of course, still calculated but subtracted directly from the appropriate $D_{i-1,j}$ to yield D_{ij}, and only the latter are entered. The row L_i can also be omitted; the $M_i - M_{i-1}$ are then obtained without recourse to S_i and L_i but by subtracting the sum of the D_{ij} not lined off in the ith column from C/d (cf. Eq. (4f)). Further shortening is possible by entering only those D_{ij} which, if checked as just described, would be ticked. This is done as follows: Calculate the D_{ij} for a given column from the bottom of the column up; whenever the result is positive, enter it; whenever it is negative (say, for $i = x, j = y$), do not enter it, cross out t_y and replace t_{y-1} by $t_y + t_{y-1}$ for all the subsequent calculations, and cross out $D_{x-1,y}$ and replace $D_{x-1,y-1}$ by $D_{x-1,y} + D_{x-1,y-1}$ for all the subsequent calculations. The resulting D's (which will not, of course, be the same as the original ones) will not need lining off. The j above the top D in each column is still s and the sum of the D's in each column still satisfies equation (4e).

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As a further test of the methods described, a programme containing 20 demands was solved. The full working is shown in Table 4; this took 30 min to fill in and a further 30 min to check fully as described, though no error was found; the whole work was done by hand. The same programme was also solved by the shortened method and the working is shown in Table 5. This took 20 min to fill in by hand, or, say, 30 min to allow for the determination of the optimum schedule and the calculation of M_n; here, however, the working originally contained an error which, although it would not have changed any of the optimal schedules, might well have escaped notice had a comparison with the checked results in Table 2 not been available.

The optimum schedule for this programme is

$$Q_{21}, Q_{53}, Q_{7,6}, Q_{11,8}, Q_{13,12}, Q_{17,14}, Q_{20,18}$$

and its (stock-holding and set-up) cost is 4942d.

Demand at Regular Intervals

One may have cases where the quantities in the demand programme are variable but the period between demands is constant (if necessary by assuming

TABLE 4. DEMAND PROGRAMME ($n = 20$): FULL WORKING

			0	0	−70	0	0	−516	−304	0	0	−30	−723	0	0	−26	0	0	0	−1104	0
								92				−30	20						44	−68	
					294		248	−10			26	158	−340			107		80	136	−172	
				318	−364	344	42	−598	−304	114	200	−110	−366		247	−133	92	235	−18	−270	
			414	−284	368	74	−208	56	154	266	−40	117	−37	345	7	116	268	63	−90	−594	66
																				210	0
$M_i - M_{i-1}$	450	36	416	82	32	368	394	296	70	264	285	450	105	196	334	90	72	378	240	384
q_i	20	12	32	8	4	16	26	30	5	11	7	23	15	14	20	6	3	9	28	24
t_i		3	10	6	2	15	12	8	6	10	21	14	7	7	4	11	9	18	6	10
		36	320	48	8	240	312	240	30	110	147	322	105	98	80	66	27	162	168	240
			96	80	24	32	390	360	40	66	70	483		98	140	24	33	81	504	144
				24		96	52			88	42	230			140		12	99	252	
							156				56	138						36	308	
																			112	
		36	416	152	32	368	910	600	70	264	315	1173	105	196	360	90	72	378	1344	384

TABLE 5. DEMAND PROGRAMME ($n = 20$) AS IN TABLE 2. SHORTENED WORKING

| $j = i-3$ / $j = i-2$ / $j = i$ | | |

some zero demands). Suppose this (constant) period is t; replace C/d by C/dt; then all the t's can be taken as unity and all the $q_i t_j$ in a column are q_i, so that the D_{ij} can be obtained directly from the $D_{i-1,j}$ by subtracting q_i. (Note that the unit of cost is then dt, so that the cost of the optimum schedule will be dt times the row total $M_i - M_{i-1}$.)

OPEN CASE

Here we suppose that the demand programme is

$$q_1, T_1, q_2, T_2, \ldots, q_i, T_i, \ldots, q_n, T_n, \ldots, q_N, T_N$$

where N may be large and the q's and T's are known for $0 < i \leqslant n$ but not for $n < i \leqslant N$.

Consider that batch (the pth) of the optimum schedule for Q_N which includes q_n. Suppose this is $Q_{ab} (b \leqslant n \leqslant a)$. Clearly nothing can be said about the $(p+1)$th and subsequent batches; it is, however, possible to draw certain conclusions about b (and, to some extent, about a) and these are in effect conclusions about the first $p-1$ batches of the optimum schedule for Q_N, since they constitute the optimum schedule for Q_{b-1}.

To avoid lengthening the paper unduly, no detailed proofs will be given for the following, except to refer to the fact that D_{ab} would be the lowest lined-off D in the ath column.

Construct a D_{ij} Table for the known part of the demand (up to $i = n$) as in the closed case and obtain m in $M_n = Z_{nm}$, i.e. Q_{nm} is the last batch in the optimum schedule for Q_n and D_{nm} is the lowest lined-off D in the nth column.

Then it can be shown that

$$m \leqslant b \leqslant n \tag{5}$$

and there are thus at most $n-m+1$ different possible batch sequences for the first $p-1$ batches in the optimum schedule for Q_N, viz., those which follow from

$$b = m, b = m+1, \ldots, b = n.$$

It is also possible to set limits on the total quantity contained in the pth batch. This is

$$Q_{ab} = Q_{nb} + Q_{a,n+1},$$

and the latter can be shown to satisfy

$$\operatorname*{Max}_{m \leqslant x < b} \left[\sum_{x+1}^{b} D_{nz} \Big/ \sum_{x+1}^{b} t_z \right] \leqslant Q_{a,n+1} \leqslant \operatorname*{Min}_{b < y \leqslant n} \left[\sum_{b+1}^{y} D_{nz} \Big/ \sum_{b+1}^{y} t_z \right]. \tag{6}$$

(6) may enable the number of possibilities to be further reduced, e.g. for $Q_{a,n+1} < 0$ or where the left side of (6) exceeds the right side. To apply (6), obtain the $(n-m)D_{nj}$ not lined off in the last (nth) column of the D_{ij} Table and also the last $(n-m)t_j$ and construct a Table as shown below (Table 6).

This has $n-m+1$ rows (and columns); the second row is headed $D_{n,m+1}$, the next $D_{n,m+2}, \ldots$, the last $D_{n,n}$; the second column is headed t_{m+1}, the next $t_{m+2}, \ldots$, the last t_n.

On the upper left to lower right diagonal (except the top left square), enter in each square the fraction of which the numerator is the D at the head of the row and the denominator is the t at the head of the column. In the upper right part of the Table (except the top row), enter in each square the fraction of which the numerator is the sum of the D at the head of the row and the numerator in the square below, and the denominator is the sum of the t at the head of the column and the denominator in the next square on the left. Find the column maxima, which are entered in the top row, and the row minima, which are entered in the last empty square in each row (i.e. that immediately on the left of the diagonal).

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Table 6 shows the working for a Demand Programme with $n = 20$; m was 15; the D_{ij} not lined off in the 20th column were, from $j = 16$ to $j = 20$ in that order, 30; 5; -10; 80; 0; and the corresponding t_j were 1; 1; 6; 2; 3.

The row minima and column maxima (shown underlined in Table 6) are the limits within which $Q_{a,n+1}$ for a given b must lie, viz., below the former but above the latter.

TABLE 6. LIMITS OF $Q_{a,n+1}$ FOR OPEN CASE

		1	1	6	2	3
		$\underline{30}$	$\underline{17\frac{1}{2}}$	$\underline{3\frac{1}{2}}$	$\underline{40}$	$\underline{16}$
30	$\underline{3\frac{1}{2}}$	30/1	35/2	25/8	105/10	105/13
5		$\underline{-5/7}$	5/1	$-5/7$	75/9	75/12
-10			$\underline{10/6}$	$-10/6$	70/8	70/11
80				$\underline{16}$	80/2	80/5
0					$\underline{0}$	0/3
	$b = 15$	16	17	18	19	20

(In Table 4 the values of b corresponding to each column are shown below that column.) Thus, for $b = 15$, $Q_{a,n+1} \leqslant 3\frac{1}{8}$; for $b = 16$, $30 \leqslant Q_{a,n+1} \leqslant -5/7$; and so on. It follows that of the six possible values of b, three can be eliminated, viz., 16, 17, 19; and we have

$$b = 15 \quad \text{for} \quad Q_{a,n+1} \leqslant 3\tfrac{1}{8}$$

$$b = 18 \quad \text{for} \quad 3\tfrac{1}{8} \leqslant Q_{a,n+1} \leqslant 16$$

$$b = 20 \quad \text{for} \quad 16 \leqslant Q_{a,n+1}.$$

Hence the first $p-1$ batches of the optimum schedule for Q_N must, in this case, be

either the optimum schedule for Q_{14} (in which case the pth batch amounts to between $Q_{20,15}$ and $Q_{20,15}+3\frac{1}{8}$);

or the optimum schedule for Q_{17} (in which case the pth batch amounts to between $Q_{20,18}+3\frac{1}{8}$ and $Q_{20,18}+16$);

or the optimum schedule for Q_{19} (in which case the pth batch amounts to not less than $q_{20}+16$).

Note:—In some cases it may be possible to eliminate some of the remaining possibilities, other than $b=m$ or $b=n$, on the ground that the corresponding range for $Q_{a,n+1}$ makes it unlikely that any q (or q's) after q_n will be within it. For example, suppose that in the case above the range for $b=18$ had been $3\frac{1}{8} \leqslant Q_{a,n+1} \leqslant 3\frac{1}{2}$, and the q's are expected to be between (say) 5 and 50; it then seems unlikely any $Q_{i,21} (i>20)$ will fall within that range and (*a fortiori*) that $Q_{a,21}$ will do so. Therefore the possibility $b=18$ might be dismissed on those grounds. But it will not be possible to eliminate $b=m$ or $b=n$ in this way, so that unless $m=n$ there will always remain at least two possible sequences for the first $p-1$ batches.

It is clear that the value of these conclusions in any given case will depend on circumstances, in particular on T_m and on when further information about future demand is likely to be available. If, for example, all the known demands would (in the closed case) be made in a single batch, $T_m = T_1$ and very little can be concluded even about the first batch of the optimal open schedule other than, perhaps, eliminating some of the $Q_i (<Q_n)$ and finding its maximum possible size $(>Q_n)$. But this would be a case where the available information about future demand is (relatively) so small that it should perhaps be treated stochastically.

In other cases it will be found that the problem is in effect solved, viz., if only a few possible values of b survive and it is found that perhaps less than $p-1$ batches need to be scheduled before further information becomes available (thereby reducing the number of possibilities still further). Consider, for example, the Demand Programme in Table 3. Applying the method just given, we find

		6	4
		$\underline{12\frac{1}{4}}$	$\underline{5}$
74	$\underline{5}$	74/6	50/10
−24		$\underline{-6}$	−24/4
	$b=9$	10	11

so that $b \neq 10$ and we have

$b=9$ for $Q_{a,12} \leqslant 5$,

$b=11$ for $Q_{a,12} \geqslant 5$.

Therefore the first $p-1$ batches of the optimal open schedule are either the optimum schedule for Q_8 or that for Q_{10}. The former is (from Table 3)

$$Q_{2,1}, Q_{8,3}$$

and the latter is

$$Q_{2,1}, Q_{8,3}, Q_{10,9},$$

so that the first two batches (covering demand up to T_8) are determined in any event. If further information is expected before the batch containing q_9 need be launched, the problem is solved.

Again, in the case worked out in Table 4, we have

		6	10
		11	$4\frac{1}{8}$
66	$4\frac{1}{8}$	66/6	66/16
0		0	0/10
	$b=18$	19	20

and hence $b \neq 19$;

$b = 18$ for $Q_{a,21} \leqslant 4\frac{1}{8}$,

$b = 20$ for $Q_{a,21} \geqslant 4\frac{1}{8}$.

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In the former case, the optimum open schedule has the batches

$$Q_{2,1}, Q_{5,3}, Q_{7,6}, Q_{11,8}, Q_{13,12}, Q_{17,14}$$

and in the latter case

$$Q_{2,1}, Q_{5,3}, Q_{7,6}, Q_{11,8}, Q_{13,12}, Q_{17,14}, Q_{19,18}$$

so that again the first 6 batches of the schedule are determined in any event and the 7th batch is not required until T_{18}.

There may, of course, be cases where the quantity which can be scheduled uniquely in this fashion is insufficient (i.e. more is required before further information becomes available) and some decision between the surviving values of b must be made without any assurance that this will be optimal. A full analysis of the problem would presumably be based on minimizing the expected loss from the wrong decision; that is to say, suppose that the surviving possibilities for b are

$$b_1, b_2, \ldots, b_j, \ldots, b_k, \ldots$$

then
$$b = b_j\left(\sum_k P_k R_{kj} = \min_j \sum_k P_k R_{kj}\right) \tag{7}$$

where P_k is the probability that $b = b_k$ and R_{kj} is the penalty for putting $b = b_j$ when $b = b_k$; but it is not proposed to develop this line in the present paper.

An alternative approach which may be used in such cases is to argue (particularly if further demand is likely to continue indefinitely) that the average (stock-holding and set-up) cost per piece for the pth and subsequent batches is

likely to be the same irrespective of which value of b is chosen. In that case one should choose that b which minimizes M_{b-1}/Q_{b-1}; e.g. for the example in Table 1, for $b = 9$, the cost per piece scheduled is $M_8/Q_8 = 731/87$; for $b = 11$, the cost increases by $194+36 = 230$ and the quantity scheduled by $15+5 = 20$, i.e. by more than 731/87 per extra piece. Hence $b = 9$ $(= m)$ would seem to be indicated. The theoretical objections to this procedure are perhaps too obvious to need exposition, but in practice it may often be adequate.

Finally, there may be cases where further demand after Q_n, while possible, may in fact not arise at all. In such cases the risk of special penalties (not allowed for in the stock-holding cost) resulting from being left with unsold stock would evidently tend to favour $a = n$ (and hence $b = m$) and the problem should in effect be treated as the closed case.

ACKNOWLEDGEMENT

The author is grateful to one of the Referees of this paper for drawing his attention to a paper by Bhatia and Garg, "Application of Dynamic Programming to a Class of Problems in Inventory Control", *J. Industr. Engng.*, **11**, 509 (1960). When this appeared, the present paper was already nearly complete and it was therefore not among the literature originally examined.

The problem addressed by Bhatia and Garg is a somewhat restricted version of the "closed case" of the present paper, the restriction being that demands occur at equally spaced time intervals. Their solution is by the equivalent of the Z-method as used in the present paper except that here the above restriction is removed and advantage is taken of result (2c) (which they do not appear to have obtained) so as to reduce the number of Z's which need to be computed.

That part of the present paper which derives the Z-method must therefore be regarded as largely anticipated by Bhatia and Garg. It was, however, felt that since the main object of the present paper is to present and discuss the D-method and since this can best be derived from the Z-method, it would be more convenient to retain the section of the paper dealing with the Z-method, and thus avoid the need for the reader wishing to follow the derivation *ab initio* to refer to another publication.

A Production Planning and Stock Control Problem in the Textile Industry

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This paper describes the techniques used in an enquiry into production planning and stock holding problems encountered by one of our member firms.

The investigation of the problem falls into three parts:

(i) The analysis of demand.

(ii) The determination of optimum stock levels.

(iii) The solution of the allocation problem which arises if the capacity available for the production of stock lines is not equal to the amount required to bring stocks back to the optimum levels. This problem has not been fully investigated, but a proposed method of solution is given in an appendix.

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INTRODUCTION

THIS paper describes an enquiry into the possibility of improving production planning methods and determining optimum stock levels for synthetic cloths and yarns, carried out for one of our member firms.

The firm manufactures a wide range of special cloths for industry. Some cloths are in regular demand, and the firm endeavours to maintain stocks of these, but others are made specially to order. Yarn is purchased from outside suppliers.

As a result of a very variable demand considerable difficulty had been experienced in determining suitable stock levels. Frequently cloths would be made for stock in anticipation of orders which failed to materialize, thus leaving substantial stocks on hand, possibly for a considerable length of time, while, in the meantime, shortages of other cloths might have occurred as a result of limitations on the amounts of productive facilities available, limitations which might be eased if a less speculative method of ordering cloths for stock could be found.

PRODUCTION PLANNING SYSTEM ORIGINALLY IN USE

Examination of the planning system revealed that production programmes were drawn up at the end of four weekly intervals and that a time lag of four weeks existed between determining a programme and putting it into effect. In compiling the programme priority would normally be given to the manufacture of cloth for customers' orders, the remaining productive capacity being used for the manufacture of standard cloths for stock. The allocation of this capacity between the different stock lines would be made on the

† Now with the English Electric Co. Ltd., Kidsgrove.

basis of estimates of future requirements supplied by the Sales Department. Stocks on hand and work in progress would, of course, be taken into account.

After drawing up a production programme, appropriate quantities of yarn would be ordered from the suppliers. Deliveries would commence roughly four weeks later and continue during the next four weeks. Roughly speaking, then, deliveries of yarn and production requirements during any period of four weeks should correspond.

Apparently the yarn manufacturers had requested that at least four weeks' notice of future requirements be given. Examination of records of yarn deliveries indicated that delays in delivery were rare, provided that the necessary advance notice was given.

POSSIBILITY OF AN ALTERNATIVE PLANNING SYSTEM

Consideration of the system outlined above suggested that the time lag of four weeks between deciding upon a production programme and putting it into effect was likely to be an important contributory cause of the firm's difficulties.

Under the existing system immediately after a programme had been determined the scheduled production for the following eight weeks would be known. An attempt ought, therefore, to be made to bring stocks up to such a level that demands likely to arise during the next eight weeks could be covered. Prior to this investigation, however, this was not possible as no detailed analysis of demand had been made which would enable appropriate stock levels to be determined. Since the necessary knowledge must be based on an analysis of demand, such an analysis was undertaken.

A more fundamental objection to the existing system was its slow response to the feed back of the latest information on demand. The elimination of the lag between the times at which a production programme was drawn up and put into effect would improve matters. For example, without such a lag, heavy demand for a particular cloth during one month, with a consequent reduction of stocks, could be corrected by an increase in production during the subsequent four weeks. Under the existing system, however, the programme for the next four weekly period would already have been determined, and corrective action would be delayed.

It is only fair to state that in practice the planning system was not as inflexible as the above remarks might suggest, as acute shortages could be remedied by deviating from the plan, but such disruptions were not encouraged and simultaneous shortages of several cloths would be difficult to deal with by such unorganized measures.

With this aspect of the system in mind, it seemed natural to suppose that under the existing system larger stocks of cloth would be necessary for a given probability of stock run-outs than would be necessary if the time lag could be eliminated.

Enquiries revealed that it would be possible to alter the production planning system if it could be demonstrated that such a change would be worth while. One obvious difficulty would arise, however, in connection with the ordering of yarn, for whereas under the existing system yarn requirements for the month in which delivery would take place would be known at the time of placing the order, this would no longer be the case if the time lag were eliminated. It would, therefore, be necessary to obtain frequency distributions from which probabilities of different levels of demand for the various yarns could be obtained. These would enable yarn stock levels to be fixed which would result in acceptably small probabilities of shortages.

ANALYSIS OF CUSTOMERS' ORDERS

In order to estimate stock levels appropriate to the two planning systems which were to be compared an analysis of demand arising during a continuous period

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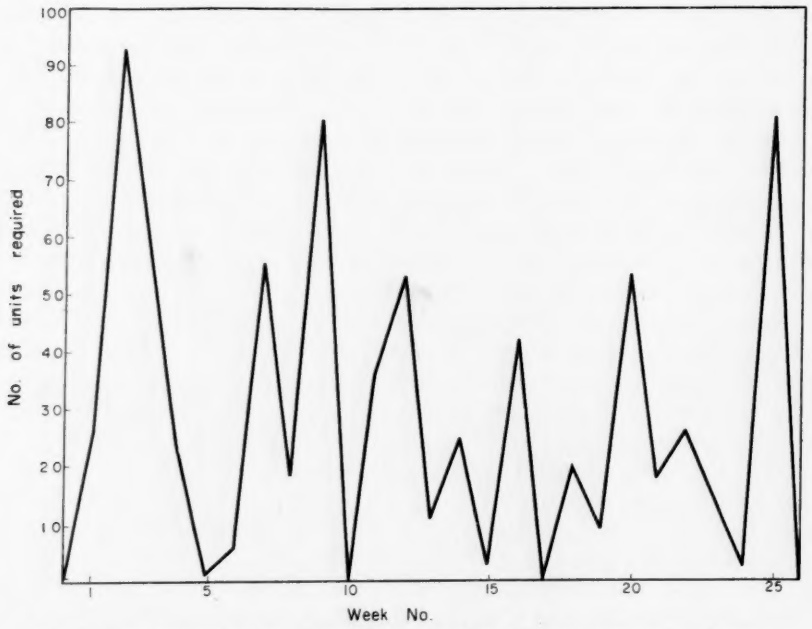

FIG. 1. *Time series showing the variation of weekly demand for one type of yarn.*

of fifteen months prior to the investigation was carried out. This involved the considerable task of recording all orders for cloth received during that period, but a smaller sampling investigation which was considered was rejected because it would not give sufficiently precise information for the many cloths and yarns involved.

Ideally we should have liked to obtain information on potential demand, taking into account losses of orders due to unsatisfactory delivery quotations arising from the existing system, but the necessary data were not available and actual orders booked were taken as representing demand.

Some of the orders, accounting for about 35 per cent of the total weight of cloth demanded, were excluded from consideration because they were placed so far ahead of the delivery time that they could be provided for in future production schedules and stocks for these were unnecessary. For the remaining part of the demand time series showing the amounts of individual cloths or yarns required during each week were plotted. These graphs, of which Fig. 1 is a typical example, revealed considerable variability from week to week, but there was no indication of trends, seasonal variation or other periodic movements. Generally speaking, demand for industrial cloth is not seasonal, but the absence of trends might have been due in part to the fact that the company tended to keep the amount of productive capacity devoted to the manufacture of these cloths roughly constant. Management is not committed to such a policy, however, and if an alteration in the planning system and in the stock holding policy resulted in changes in the level of demand, then alterations could be made in the capacity available for the manufacture of these cloths.

To make a comparison between the two planning systems under consideration it was necessary to determine distributions of (a) the amounts of standard cloths demanded in a period of four weeks, (b) the amounts of different yarns required in a period of four weeks for all types of cloth, and (c) the amounts of standard cloths demanded in a period of eight weeks. (a) and (b) would be required for plans without a time lag and (c) for an improved form of the existing planning system. In order to facilitate the construction of cost or profit functions at a later stage of the investigation it was decided to specify the distributions where possible by theoretical formulae, rather than numerically. Theoretical formulae also have the advantage of graduating the data.

At first, attempts were made to relate the quantities demanded in various fixed periods of time to the arrival rate of customers' demands. As demands arrived at random it was thought that a suitable representation of the probability of receiving demands for more than X yards of a particular cloth in time T would be given by the formula

$$G(X) = e^{-\lambda T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} \int_X^{\infty} f_n(x)\, dx$$

where λ is the arrival rate of orders, and $f_n(x)$ is the n-fold convolution of $f(x)$, the distribution of amounts of cloth demanded on individual orders. However, it did not prove possible to obtain formulae to represent the distributions of amounts demanded per order in a satisfactory manner, and consequently this approach had to be abandoned.

As an alternative, the distribution of the total amount of a particular yarn or cloth demanded in a fixed length of time was considered. With records of orders received extending over a period of only fifteen months it was not possible to consider the amounts demanded in successive periods of four or eight weeks and attempt to fit theoretical distributions to these, as such a procedure would result in far too few observations becoming available. Instead, attempts were made to fit theoretical distributions to the observed weekly demands, and to use the properties of characteristic functions to obtain probability distributions of demand in longer periods.

It was found that in many cases the Gamma distribution,

$$f(x) = \frac{a^p}{\Gamma(p)} e^{-ax} x^{p-1},$$

where X is the length of cloth or weight of yarn demanded in a week, provided a good fit to the data. The parameters a and p were estimated from the first two moments of the observed distribution. The distributions of demand in periods of four and eight weeks were obtained by making use of the fact that the sum of a number of Gamma variates is also distributed in the Gamma form.

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In some cases where it was not possible to obtain a good fit by the use of the Gamma function it was found that the observed cumulative distribution could be represented very well by the mixed exponential distribution

$$F(x) = \int_x^\infty f(x)\,dx = a\,e^{-\lambda x} + b\,e^{-\mu x} \quad (b = 1-a).$$

The parameters of this distribution were estimated by equating three points on the observed cumulative distribution to corresponding points on the theoretical curve and solving for the three parameters. The distributions of sums of variables distributed in this form may be determined by the use of characteristic functions or Laplace transforms. The Laplace transform of x is

$$[a\lambda/(\lambda+\theta) + b\mu/(\mu+\theta)],$$

and if we let the amount demanded in a period of n weeks be denoted by the variable y the transform of y is

$$\sum_{j=0}^{n} \binom{n}{j} \left(\frac{a\lambda}{\lambda+\theta}\right)^{n-j} \left(\frac{b\mu}{\mu+\theta}\right)^{j}.$$

The frequency function of y may be obtained by inversion, i.e.

$$f(y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\theta y} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{a\lambda}{\lambda+\theta}\right)^{n-j} \left(\frac{b\mu}{\mu+\theta}\right)^{j} d\theta.$$

Evaluation of this expression by contour integration results in an unattractive sum of Gamma functions which is difficult to evaluate by hand computation.

However, computational problems were easily overcome by making use of the Mercury electronic computer installed at the University of Manchester.

In order to make use of the properties of characteristic functions it was necessary to ensure that the amounts demanded in successive weeks were independent of each other. From an examination of some of the time series of weekly demands, this appeared to be the case, but for a sample of cloths and yarns a more exact test was made by comparing the number of turning points in the observed series with the expected number of turning points in a completely random series. No significant differences were found.

RULES FOR RE-ORDERING

Having obtained the appropriate probability distributions it was possible to construct tables showing the stock levels associated with various risks of stock run-outs, and to recommend the following simple re-ordering rules.

(a) For stocks of standard cloths, using a planning system with no time lag between drawing up the production programme and putting it into effect, $M_1 - x$ units should be ordered, where M_1 is the stock level corresponding to an acceptable risk of shortages and x is the number of units in stock.

(b) Under the existing planning system with its time lag, it would be necessary to provide for possible requirements for eight weeks subsequent to making the plan. This would result in the rule: order $M_2 - x - y$ units where M_2 is the quantity given in the table relating to demand in a period of eight weeks and corresponding to the chosen risk of running out of stock, x is the present stock level, and y is the quantity included in the previous production programme, but not yet manufactured.

(c) The amount of yarn which should be ordered when using the first planning system is $(M_3 + x - y - z)$ units where M_3 is a stock level obtained from the table of yarn stock levels corresponding to given probabilities of running out of stock, x is the amount of yarn required to meet the production programme for the next four weeks, y is the amount of yarn on order from the suppliers for delivery during the same period and z is the amount of stock on hand.

It will be observed that in recommending the above rules no account has been taken of delays in delivery which might occur within the period in which the cloth is being produced. For example, rule (a) is designed to ensure that M_1 units become available during a planning period, not that the stock level be brought up to M_1 units at the beginning of the period. This is thought to be satisfactory as demands occur at random throughout the month and not all at the beginning of a month. If difficulty were experienced with any particular cloth it is likely that appropriate action could be taken by the Progress Department.

In a sense rule (b) is inferior to rule (a) for although by choosing a base stock level of M_2 units it is possible to ensure an acceptably low risk of running out of stock during a period of eight weeks considered as a whole, a high level

of demand in the first four weeks might result in a risk greater than that desired being run in the second half of the eight-week period, whereas rule (a) ensures that a known risk of run-outs is taken each month.

However, even with the existing planning system it would be possible to ensure that, with known probability, sufficient cloth would be available during the second half of the eight-weekly period to ensure a risk of run-outs less than or equal to the desired level. To do this one observes that the stock level at the end of each four-weekly period is a regular Markov chain, whence it can be shown that the probability of the amount of stock becoming available during the second half of the period of eight weeks being M_1 units or less is equal to

$$\frac{F(M_2'-M_1)[1-F(M_1)]}{1-F(M_2'-M_1)F(M_1)},$$

where

$$F(y)=\int_y^\infty f(y)\,dy$$

is the distribution function of demand in a period of four weeks, M_2' is the new base stock level to be used in conjunction with rule (b) and M_1 is the amount of stock which should become available during the second half of the period if the desired risk of running out of stock is not to be exceeded. In order to make use of this theory it is necessary to assume that orders which cannot be met out of the amount of stock becoming available in a period of four weeks will be lost. This is not realistic, but it is probably not a very important factor when dealing with low risks of shortages.

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The values of M_2' obtained in this manner were very high compared with the values of M_1 used in conjunction with rule (a) and were about 50 per cent higher than the values of M_2 used in conjunction with rule (b). This is no doubt largely because these levels gave assurance of at least M_1 units becoming available in any period of four weeks, rather than an assurance of exactly M_1 units becoming available.

Even if the existing planning scheme were looked upon in its most favourable light (from the point of view of stock holding costs) by taking as base stock levels the amounts M_2 used in connection with rule (b) it became apparent that the elimination of the time lag in the production planning would result in considerable savings in stock holding costs, average stocks being, in general, about 60 per cent higher with the existing planning scheme for a given probability of shortages than would be the case with the alternative scheme.

For a few types of yarn or cloth it was not found possible to find satisfactory formulae describing the observed distributions of demand, and the operation of rules (a) and (b) was simulated over artificial periods of about 2000 weeks with a range of values of M_1 and M_2. As might be anticipated from the results of closed loop control theory, the simulations showed that the variance in stock levels could be considerably reduced by eliminating the time lag in the response

of production to changes in demand. The large variances occurring under the existing system were a good indication of the source of the firm's difficulties.

RULE (a)

Value of M_1	Percentage of weeks in which a deficit occurred	Average surplus	Average deficit	Variance of positive stock level	Variance of negative stock level
25	2·2	16·50	0·07	45·22	0·64
30	0·5	21·453	0·03	47·02	0·21
Value of M_2		RULE (b)			
25	8·6	22·49	0·57	238·30	5·89
30	4·5	27·17	0·25	254·10	2·20

Even after taking yarn stocks into account, comparison of the two systems from the point of view of stock holding costs was sufficient to convince management that a change from the existing system towards the proposed alternative, whereby production programmes would be put into effect as soon as possible after they had been drawn up, was justified. Subsequent work on the project, which was mainly concerned with the determination of the risk of shortages which should be taken, was carried out on the assumption that such a change would be made.

DETERMINATION OF OPTIMUM STOCK LEVELS

In this part of the enquiry an attempt was made to determine stock levels which would result in minimum costs or maximum profits, when costs associated with both stock holding and shortages were taken into account.

As the necessary calculations were quite extensive, it was decided to investigate only a sample of cloths, in the hope of reaching a general conclusion about the risk of shortages associated with least cost solutions. The risk of a shortage was defined to be the probability that demands in a month exceed the amount of cloth becoming available during that month. As previously indicated, we were not concerned with temporary shortages which might occur within the month.

A well-known cost function providing a reasonably good model for the situation which would exist under the proposed new planning system is

$$C(M) = c_1\int_0^M \left(M-\frac{y}{2}\right)f(y)\,dy + c_1\int_M^\infty \frac{M^2}{2y}f(y)\,dy + c_2\int_M^\infty \frac{(y-M)^2}{2y}f(y)\,dy$$

where $C(M)$ = total expected cost associated with a base stock level of M units (M has previously been denoted by M_1, but as we shall now be dealing with only

one planning system, the suffix has been dropped.)

c_1 = stock holding cost of one unit of cloth for one month.

c_2 = cost per month of a shortage of one unit of cloth.

$f(y)$ = frequency function of demand in a period of four weeks.

When using this function it is convenient to work in units of c_1, the expression for the total expected cost then becomes a function of the ratio c_2/c_1. This

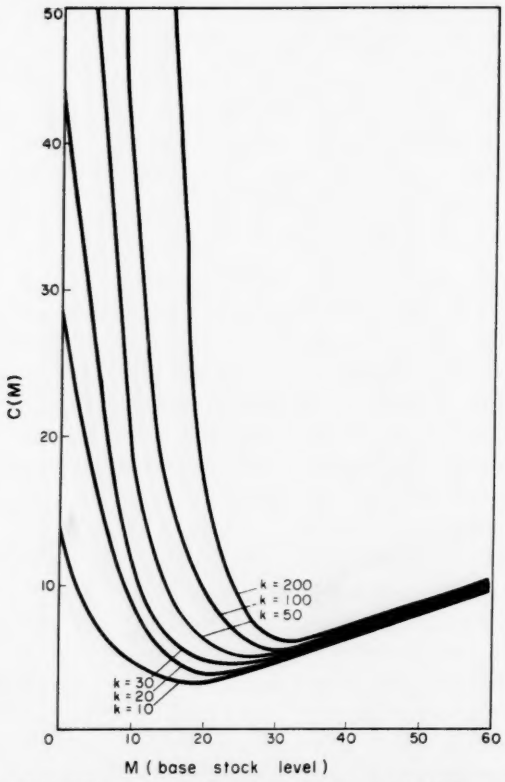

FIG. 2. *Cost curves plotted for several values of the ratio* $k = c_2/c_1$. *The cost,* $C(M)$, *is in units of C, multiplied by a scale factor.*

model assumes (a) that, on average, demand will occur uniformly throughout the month and (b) that the stock level will be brought up to M units at the beginning of the month. Assumption (a) is reasonable, but (b) is palpably not true. However, from the manner in which shortages have been defined it follows that the expected cost associated with shortages will not be affected by this assumption, but stock holding costs will tend to be overestimated.

By differentiation of $C(M)$ it can be shown that for a least cost solution a value of M should be chosen which will make the risk of a shortage equal to

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$c_1/(c_1+c_2)$. This did not prove to be very helpful, however, for, as must often be the case in this type of investigation, considerable difficulty was encountered in estimating c_2, involving as it does the question of loss of goodwill associated with shortages. For a sample of cloths, however, the cost function was graphed for various values of the ratio c_2/c_1 in order to obtain information on the general shape of the curves and to find out how sensitive were optimum stock levels to changes in the value of this ratio.

A typical set of cost curves is shown in Fig. 2. For high values of c_2/c_1 a value of M of about 30 units, corresponding to a risk of shortage of 0·10 is optimum. For lower values (it is unlikely that a value lower than 20 would be given to this ratio), a base stock level of 23 units, corresponding to a shortage risk of 0·20, would be appropriate. However, even for the lower values of this ratio the expected costs involved in using a base stock level of 30 units would not be much higher than those associated with a level of 23 units, hence for this cloth we should feel justified in recommending a base stock level of 30 units.

Our experience indicates generally that the location of the minimum point on such curves is not very sensitive to changes in the value of c_2/c_1, but such changes may alter the general shape of the curve quite considerably. This would not be very important for a business ordering material from an outside supplier, or for a manufacturer with virtually unlimited capacity, as sufficient could always be ordered to bring stocks back to the optimum levels. In the situation with which we are dealing, however, sufficient productive capacity may not be available to enable this to be done (it will be recalled that the proportion of the firm's capacity which can be devoted to stock lines is largely dependent on the volume of cloth being produced to order). If insufficient capacity is available one is faced with an allocation problem, the answer to which might be somewhat sensitive to changes in the value of c_2/c_1.

Apart from the difficulty of estimating c_2, it may be doubted whether it is correct to assume that a constant cost can be associated with every unit by which demand exceeds the amount available, independently of the size of the deficit. In practice this assumption is unlikely to be true as some customers will wait for delivery, so that a zero cost can be assigned to some shortages, while others will refuse to place orders if confronted with a delay in delivery. The proportion of customers who will wait is likely to depend on the delay time, that is, on the average deficit, so the value of c_2 should be a function of the size of the deficit, rather than a constant.

With this point in mind experiments were made with a method proposed by Hanssmann (*J.O.R.S.A.*, 1959) of determining stock levels which maximize expected net profit. The expected net profit is the difference between profits on the expected volume of sales and the stock holding costs incurred in holding enough stock to achieve that level. The method is based on the hypothesis that delays in delivery result in some potential customers obtaining their requirements elsewhere, and the greater the average delay, the greater will be the

proportion of customers who do this. With each proposed stock level there will be associated a certain average delay and a corresponding proportion of lost customers.

It is assumed that the potential demand for a cloth follows a known distribution with mean m and standard deviation σ, but that actual demand amounts to only a proportion, $a(t)$, of this depending on the delay time t, associated with the stock holding policy in use. According to our assumptions the expected demand during a month is $m(t) = a(t)m$. The standard deviation of monthly demand is assumed to be $\sigma(t) = a(t)\sigma$, it is further assumed that the demand distribution retains the same functional form despite changes in the value of t, the delay time.

Denoting the expected deficit during a month by $D(t)$ we have, since the production rate per month is $m(t)$, an average delay time of

$$t = \frac{D(t)}{m(t)} = \frac{D(t)}{a(t)m}.$$

$D(t)$ can now be determined, given t and a knowledge of the function $a(t)$.

It is now necessary to determine a base stock level which will give an average deficit of $D(t)$. This may be done by solving the equation

$$\int_{M(t)}^{\infty} [y(t) - M(t)] f[y(t)]\, dy(t) = D(t)$$

for $M(t)$, where $M(t)$ is the required stock level, and $f[y(t)]$ is the frequency function of the amount demanded in a four-weekly period when the average delay time is t.

Having ascertained $M(t)$ for different values of t, the following profit function associated with the stock holding policy may be constructed:

$$P(t) = pm(t) - c\bar{M}(t)$$

where p is the profit per unit of cloth, c is the unit holding cost for one month and $\bar{M}(t)$ is the average stock level.

The expected profit may now be plotted as a function of the delay time, or, as there is a unique relationship between average delay time and the required base stock level, $M(t)$, the delay time can be eliminated and $P(t)$ can be plotted as a function of $M(t)$. The optimum stock level is then obtained directly.

Lack of data caused some difficulty in the estimation of $a(t)$. However, in this particular study the shape of this function proved rather less difficult to estimate than values of c_2. From opinions expressed at the firm it was possible to determine values between which the function would be likely to lie for a few values of the delay time t. It was then possible to graph two functions, $a_1(t)$ and $a_2(t)$, representing extreme assumptions about the effects of delays in delivery, and two corresponding profit functions. In carrying out this procedure we were investigating the possibility of there being base stock levels which would

be nearly optimum on either assumption. It did in fact prove possible to find such levels for all cloths in the sample investigated.

Although estimates of stock holding costs could be made for each cloth it was considered desirable to graph the profit functions for more than one holding cost, as such costs might change with time and, in any case, we were interested in the effect of such changes on the optimum stock level. Two sets of profit curves are illustrated in Figs. 3 and 4. Stock holding costs are expressed as percentages of the selling price of cloth; the percentages might appear rather low, but this is justifiable, as holding costs should be based on the lower

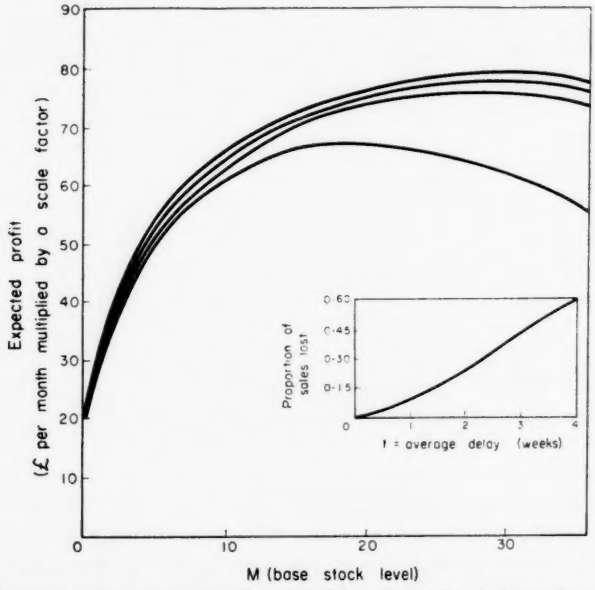
Expected profit
(£ per month multiplied by a scale factor)
M (base stock level)
Proportion of sales lost
t = average delay (weeks)

FIG. 3. *Profit functions based on annual stock holding costs of* 4, 5, 8 *and* 24 *per cent of the selling price. Inset is a graph of the function* $1 - a(t)$.

manufacturing cost rather than on selling price. The three upper curves represent profit functions associated with stock holding costs likely to be incurred in financing stocks by means of a bank loan. It seems reasonable to take the view that holding costs are closely linked with the cost of borrowing money from a bank, because unlike fixed assets, stocks can often be realized without appreciable loss, and for this reason a rate of return only slightly above the marginal cost of obtaining a loan should be required on investment in stocks.

Within the range of variation covered by the upper three curves changes in stock holding costs appear to have little effect on optimum stock levels. It was important to establish this point as attempts might be made to reduce the effects of a rise in interest rates by cutting down on stocks. However, the adverse

effects of increasing the probability of shortages predominate, and our calculations indicate that it is hardly worth while making alterations in stock levels because of a moderate change in the rate of interest.

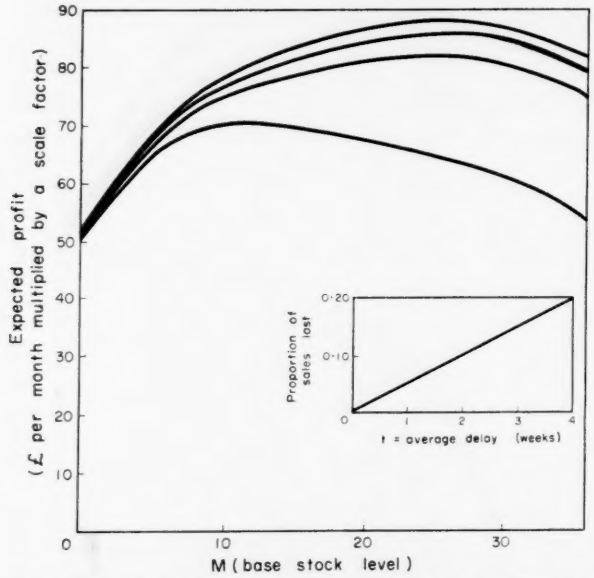

FIG. 4. *Profit functions based on the less severe assumption about the effect of delays. Inset is a graph of the function* $1 - a_2(t)$.

Although the rate of interest has been taken as a basis for evaluating stock holding costs, other views on the matter exist. One is that the rate of return on stocks should be the same as that on other forms of investment. Then the monthly carrying cost of a unit of stock might have to be regarded as, say, 2 per cent of the selling price. Such an assumption would result in considerably smaller stocks being held than if the rate of interest were used as a basis of evaluating holding costs, as is illustrated by the lower curve on Figs. 3 and 4.

Examination of the two sets of curves in Figs. 3 and 4 reveals that if the rate of interest is taken as a basis of evaluating stock holding costs, it is not difficult to determine an optimum base stock level, as on each curve the maximum occurs at a point corresponding to a level of about thirty units. This represents a risk of shortages of about 0·10, the same result as was achieved by the use of cost functions.

For all types of cloth investigated in this manner it was found possible to determine stock levels which were very near to the optimum which ever assumption was made about the form of the function $a(t)$. Furthermore, although the

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optimum risk of shortages was not the same for all cloths it was found that the adoption of a shortage risk of 0·10 would not lead to costly deviations from the maximum profit level; hence, on the basis of the results obtained for the sample of cloths investigated, it was decided to recommend that a risk of a shortage occurring within a month of 0·10 for all cloths be taken.

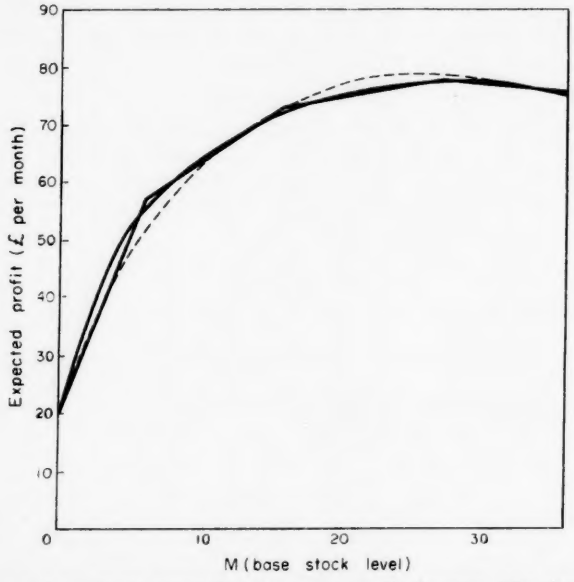

FIG. 5. *Linear and quadratic approximations to a profit function. The dotted line is the quadratic approximation.*

Similar investigations were not carried out in connection with stock levels for yarn as it was decided to carry sufficiently large stocks to enable a low risk of shortages to be run.

DETECTION OF CHANGES IN THE LEVEL OF DEMAND

Although no time trends were apparent in the analysis of demand it could not be assumed that trends would not appear at some time. In particular a change in the inventory policy could result in an alteration of the level of demand. In order to detect such changes, arrangements were made for the firm to collect information on demand and to plot it on control charts as a routine. It was recognized that such charts would not be very sensitive instruments, but at least they should enable substantial changes to be detected, and make possible the determination of new demand distributions.

CONCLUSIONS

It is too early to estimate the effects of our proposals, but the stock levels recommended have been accepted as the basis of a new stock holding policy. Arrangements for a gradual change over from the existing planning system to the proposed system have been made, and partly put into effect.

The proposed stock levels differed substantially from those which had often been aimed at, being perhaps only half as high for some cloths, but higher than existing levels for others.

ACKNOWLEDGEMENTS

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APPENDIX

Allocation of capacity to the production of different types of cloth

As the ability of the firm to implement the rules for the determination of the amount of cloth to be produced is dependent on the amount of capacity available for the production of stock lines, it is apparent that on occasions insufficient capacity might be available to bring stocks up to the 0·10 risk level. On the other hand, more capacity might be available than is required for this purpose, but management might feel that it ought to be used in order to avoid, for example, the loss of trained operatives. In either case an allocation problem exists.

This problem has not been investigated in great detail as it is felt that a reasonably good allocation can be made by using the tables of stock levels associated with different risks of shortages, and studying the general behaviour of the profit functions computed for a sample of cloths. If a more accurate answer is required, however, programming methods can be used, and for a hypothetical case, involving only those cloths for which profit functions have been computed, this has been done. The method used was that of approximating profit functions by second-degree curves, and maximizing the resulting quadratic form subject to linear constraints on the number of machine hours available at each process.

Stated in algebraic terms the problem is to maximize

$$C = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} x_i x_j$$

where x_0 is defined to be unity subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i \quad (i = 1, 2 \ldots m)$$

where C is the expected profit, x_j is the amount of product number j to be produced, a_{ij} is the number of machine hours of process number i used in making a unit of product number j, b_i is the number of machine hours available at process number i, m is the number of processes and n is the number of possible products.

In the trial case considered profit functions associated with the more severe assumption about the effect of delays on the volume of trade were used, and it was assumed that insufficient capacity was available to enable stocks to be brought back to the levels giving a 10 per cent risk of shortages. The results gave a worth-while improvement in the value of the objective function as compared with the solution proposed by the firm based on their conventional methods. The calculations were carried out on the Mercury computer. At the present time the computer programme written for the solution of quadratic programming problems does not automatically provide for the use of an artificial basis technique. Such a technique would be necessary if more capacity were available than was required to maximize the objective function, but it was thought advisable to use this capacity even though expected profits would be reduced as a result of increased stock holding costs. However, the desired result may be obtained by attaching large penalty costs to one or more of the slack variables and including these variables in the objective function.

The results obtained from this experiment are promising, but more work will have to be done to obtain information on the sensitivity of answers to alterations in the form of the function $a(t)$, the "decay function" of sales. In this particular study it proved much easier to fit second-degree curves to profit functions than to the cost curves which were also computed.

An alternative method of solving this allocation problem would be to convert it into a linear programming problem. This may be done by approximating the profit function for each cloth by sets of straight lines. The linear and quadratic approximations to the profit function for cloth number one are illustrated in Fig. 5. Using linear programming it is necessary to associate more than one variable with the production of one type of cloth, the actual number used being equal to the number of straight line sections used in obtaining an approximation to the profit function. Limitations must, of course, be imposed on the values which these variables can take. Thus from Fig. 5 we see that

$$x_1 \leqslant 6 \quad x_2 \leqslant 10 \quad x_3 \leqslant 10, \quad \text{etc.}$$

where x_i is the amount of cloth number one which is to be produced in the region covered by the ith straight line section of the approximation to the profit function.

Linear programming may be easier to use than quadratic programming as the necessity of fitting second-degree curves is avoided. In this particular application it is not possible to fit curves once and for all to the whole range of the function, as, if some stock is on hand when the production for the next four weeks is being planned, only a portion of the curve will be of interest to the planner, and a curve must be fitted to this portion. The same difficulty does not arise if linear programming is used provided that appropriate restrictions are placed on the values which variables associated with the production of one particular cloth can take. If linear programming is used it might be possible to make use of the dual solution to obtain information on "opportunity costs" associated with departures from the optimum schedule, and the value of additional capacity at bottleneck processes. These values would differ from one planning period to another, depending on the amount of capacity available for the production of stock lines and on the opening stock levels, but persistent bottlenecks could be noted and the value of additional machinery estimated. Against these advantages of linear programming must be offset the increased size of the problem which results from associating several variables with the production of one type of cloth.

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A Stock Control Policy for Important Spares in a Two Level Stores System

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(National Coal Board)

This paper describes the results of an investigation to determine correct re-order levels and order quantities. The problem had three important features:

(i) a simple stock control system was required for about 15,000 items;
(ii) run-out costs needed to be estimated;
(iii) the solution required best stock levels and order quantities for a central store and for smaller stores supplied by it.

The problem arose because it was felt that production delays were being caused by the stores too frequently being unable to supply spares.

Items were grouped by their importance and portability. Average run-out costs were estimated for each group. An approximate model of the two level stock holding problem was derived. An iterative process was required to obtain solutions.

It was observed that the stock control rules were insensitive to run-out cost, and that items of equal run-out cost had widely differing run-out probabilities. This indicates that, in this case, it is invalid to group items by importance and to assign fixed run-out probabilities to each group.

1. INTRODUCTION

THE work described in this paper is concerned with the supply of certain important spare parts in one Area of the National Coal Board. The Area has stores at 16 collieries and an Area central store. The problem was to find the best levels of stock holding for individual spares at each store in order to reduce to a minimum the total costs of stock holding and failure to supply when needed.

2. STATEMENT OF THE PROBLEM

The problem was concerned with spares for machines which, if they break down, hold up production. If spares are not available when and where required the loss is greater than that caused by the time taken to effect a repair. The department responsible for production were conscious of this high cost and wanted to know whether stocks of such items were correct.

The spares are bought by a central purchasing office and delivered to the central store. From there they are issued in smaller quantities to the collieries. A regular daily delivery service supplies routine colliery needs, and emergency demands for spares are met by special lorry deliveries from the central store.

A re-order level system (without optimal policies) was already in use, and preferred by the local staff for practical reasons. Cyclical review was estimated to have no advantage by theoretical analysis and that possibility was not further considered.

† Now with Richard Thomas and Baldwins Ltd.
‡ Now with I.C.I. Ltd.

3. APPROACH TO THE PROBLEM

The problem was tackled by measuring the cost of failing to supply an item immediately and the cost of stock holding, and using these to determine best re-order levels for each store. The first cost required a study of the way in which the spares were used. A random sample of 300 items was selected from a total of 15,000. Details of stocks, prices, delivery and patterns of consumption were gathered for each item. Technical information about the way in which items failed was obtained from local engineers. They classified items as portable or not and stated whether or not an item would cause a hold-up when it failed.

There are extreme circumstances when even the most trivial spare can hold up production and cause great loss. However, it would be foolish to hold stock to guard against the highest possible cost of run-out occurring. An estimate of the expected cost of run-out was required which took into account the frequency of different consequences of failure to supply. These frequencies are difficult to assess from records because there is seldom enough history of consumption and where there is sufficient consumption there is no consistent record of the consequences of failure to supply. We were satisfied that when a particular spare is used it falls into one of two categories mentioned above. Either most of its consumption is used for breakdowns (priority items) or hardly any of its consumption is so used (non-priority items).

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4. COSTS

4.1. *The Factors affecting Cost*

For the type of spares under consideration the cost of running out of stock depends on several things. For example, for a priority spare the cost is almost entirely loss of revenue from decreased production caused by a failure to supply. This depends on:

(i) The machine, since different machines have different production rates.

(ii) The distance of the machine underground from the store, since this will affect the time taken to supply.

(iii) Whether or not the item is portable, since this also affects the time taken to supply.

(iv) What alternative useful work the men concerned can do while the machine is out of action.

(v) How near the end of a shift it is when the breakdown occurs.

(vi) The value of a unit of product.

(vii) The amount of useful work that can be done on the breakdown while waiting for the spares to arrive from the store.

It would be possible to determine all these factors, but the calculation would be different for every spare and the results would be too complicated to use. To derive rules of general application, averages have been used where possible.

4.2. *The Length of a Stores Delay*

When a machine breaks down underground, a fitter will normally inspect the machine for faults. At some time in his examination he will be able to identify the spares, if any, needed for the repair. He will then telephone the surface to explain his requirements. This starts a train of three operations on the surface:

(a) The items needed must be identified by part numbers and the request either met at the pit or telephoned to the central store. This takes about a quarter of an hour.

(b) The items requested at the central store must be found and delivered to the pit-top. In normal conditions this takes about half an hour for any pit in the Area. Delivery from another pit or from outside the Area takes longer.

(c) The item must be taken underground. The time that this takes depends upon whether the item is portable or not. If it is portable, then it will take about one hour for a man to carry it to the face. If the item is not portable, it must be taken in a small rail wagon and this takes about four hours.

The sum of these three times is then the stores service time, but this is not necessarily all lost time. The fitter may continue stripping the machine in order to be ready to use the spares when they arrive. The time usefully spent in this way we call the stripping time, and use the difference between the stores service time and the stripping time to estimate the delay waiting for spares. The average stripping time was 20 minutes for priority portable items and 100 minutes for priority non-portable items.

4.3. *The Effective Length of a Stoppage*

It was thought possible that some of the production lost during a delay could be made up by later work. Analysis of previous time studies showed, however, that loss of production due to any one cause of stoppage is approximately equal to the working time lost due to that cause of stoppage, multiplied by the average rate of production.

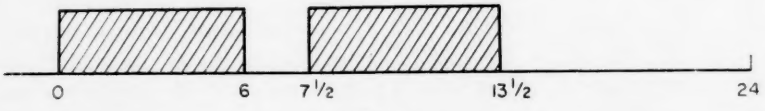

FIG. 1.

The working time lost during a breakdown of length T is, on the average, less than T because of the chance of a breakdown occurring near the end of the shift and extending into idle time. The exact amount depends upon the lengths of, and the intervals between, shifts. In this Area the usual times for working

are two six-hour shifts separated by an hour and a half, and followed by ten and a half hours' idle time each day. We assume that a breakdown is equally likely to happen at any time during working time. The frequency of failure may be drawn as in Figure 1.

The useful time that is lost is that proportion of the stoppage that occurs during a shaded period in the graph. The expected loss of working time T' during a stoppage of length T is shown in Figure 2.

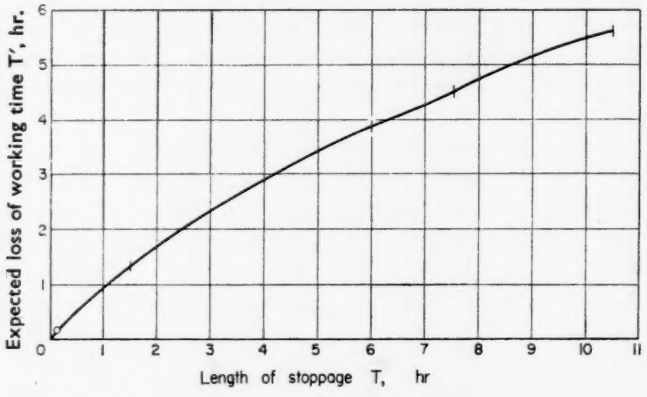

FIG. 2.

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4.4. *Costs due to Stores Delays*

By using records of breakdowns and estimates given by engineers it was found that when a breakdown occurs to a piece of underground machinery:

(a) For the average portable item, detection of the trouble took twenty minutes, stripping took twenty minutes, and re-assembly twenty minutes.

(b) For the average non-portable item, detection took twenty minutes and stripping and re-assembly took 100 minutes each.

At the pit store the spares needed must be identified by part numbers. The spares are found by looking first in the pit store, then in the central store and so on until the demand is satisfied. The spare must then be transported from its store to the pit-top, and from the pit-top to the breakdown. Using estimates given by underground staff and stores staff, we concluded that the times taken for this service are as shown in Table 1.

The figures in the table give the average value of production lost (i.e. expected loss of working time × value of marginal production) in the event of stores at the different levels being unable to meet an urgent demand. For example, a demand for a portable item which is eventually met outside the Area causes a stores delay costing £135.

TABLE 1. COSTS DUE TO STORES DELAYS IN THE EVENT OF A BREAKDOWN (PRIORITY ITEMS)

Store eventually supplying	Stores service time (min.)				Length of stoppage (min.) Detection + max. (stripping, stores service) + re-assembly	Expected loss of working time (min.) from graph	Cost of stores delay (£)
	Identifying and locating part in store	From supplier to pit-top	Travel underground	Total			
Portable items							
Pit store	15	0	60	75	115	98	0 (Datum)
Central store	15	30	60	105	145	118	20
Another pit	15	60	60	135	175	138	40
Outside area	15	240	60	315	355	231	135
Non-portable items							
Pit store	15	20	240	275	395	245	0 (Datum)
Central store	15	30	240	285	405	250	5
Another pit	15	60	240	315	435	263	20
Outside area	15	240	240	495	615	332	85

4.5. *The Cost of Run-out of Portable Priority Items*

The cost of a run-out at a pit store, when a priority item is required, is mainly the loss of revenue from the lost production. The cost of the extra transport required is comparatively negligible. If the item is brought from the central store the run-out costs £20, if it comes from another pit it costs £40, and if it comes from outside the Area the cost is £135. When using these costs to determine stock levels for pits, we took, as a first step, the probability of the last two costs being incurred as zero. This gave the average cost of run-out at the pit as £20. Later, the run-out probabilities at the central store, given by levels we determined, were shown to be small enough to justify the approximation.

The means of estimation of the average cost of run-out at the central store is outlined in 5.2.3, and the detailed analysis given in an Appendix. The average run-out cost for portable priority items at the central store was £2.

4.6. *The Cost of Run-out of Non-portable Priority Items*

As before, the cost of run-out of non-portable priority items is the loss of revenue, and in using the run-out costs to determine levels for pit stores we estimated that the probability of a run-out at the central store was zero. This gave the average cost of a run-out as £5. Since non-portable items usually cost at least £10 per year to hold and rarely have high usage at a pit, it was concluded that non-portable priority items should not be held at pits.

The average cost of run-out of a non-portable item at the central store was then £85 and this cost gave levels which justified the assumption of a very small probability of failure to supply at the central store.

4.7. *The Cost of a Non-priority Run-out*

The cost of a run-out for a non-priority item depends upon the action the stores department takes to replenish the item. There will be more work at the central store and some little dislocation of activity at the pit. We estimate that these cost 5s. If the central store cannot supply then the run-out may become more serious. We estimate that the average cost of run-out of a non-priority item at the central store is £2.

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4.8. *The Costs of Holding and Ordering*

The marginal cost of holding an item for a year was found to be 10 per cent of its price. The marginal cost of ordering was estimated as 7s.

5. ANALYSIS

In this work the following symbols will be used:

- q order quantity,
- d_i annual demand at pit "i",
- D annual demand in the Area,
- p price of an item,
- l lead time,
- h the fraction of the price of an item which is the cost of holding it for a year,
- m_i the re-order level at pit "i",
- M the re-order level for the Area,
- r_1 the average cost of a run-out at a pit (i.e. per unsatisfied demand),
- r_2 the cost of a run-out when pit and the central store cannot supply,
- R the average cost of running out at the central store (i.e. per unsatisfied demand),
- t the time taken to replenish a pit store from the central store.

5.1. *Best Order Quantities*

(a) *Pit stores.* It was found that the number of staff would not be altered by a small change in the frequency of orders for reprovisioning pits from the central store. The re-order quantity for pits was fixed at either one item or one week's supply, whichever was the greater.

(b) *Central store.* For orders to suppliers from the central store the standard formula for order quantity was used. Using holding cost of 10 per cent and

7s. ordering cost, this gave

$$q = \sqrt{\left(\frac{7D}{p}\right)}.$$

5.2. *Best Re-order Levels*

5.2.1. *General analysis.* For items with a Poisson demand distribution the average cost of holding and running-out of stock for a re-order level is given by

$$C_M = R\frac{D}{q}\sum_{s=M+1}^{\infty}(s-M)\frac{(lD)^s}{s!}e^{-lD} + \left(\frac{q+1}{2}+M-lD\right)hp.$$

This is a good approximation when lD is small compared with M.

The difference between the cost of using level M and level $M+1$ is

$$C_{M+1}-C_M = hp - R\frac{D}{q}\sum_{s=M+1}^{\infty}\frac{(lD)^s}{s!}e^{-lD}.$$

The boundary between holding M and $M+1$ occurs when $C_M = C_{M+1}$, that is,

$$\frac{hpql}{R} = lD\sum_{s=M+1}^{\infty}\frac{(lD)^s}{s!}e^{-lD}.$$

A chart can now be drawn up to determine the best re-order level for any given values of the parameters lD and $hpql/R$, all of whose components will be known for an item.

For items with a high demand the demand distribution is observed to be normal with standard deviation equal to half the mean. Here the value of C_M is given by

$$C_M = R\frac{D}{q}\frac{2}{lD\sqrt{(2\pi)}}\int_{M+\frac{1}{2}}^{\infty}(x-M)\exp\left\{-2\left(\frac{x-lD}{lD}\right)^2\right\}dx + \left(\frac{q+1}{2}+M-lD\right)hp.$$

The boundary between holding M and $M+1$ occurs when

$$\frac{hpql}{R} = \sqrt{\left(\frac{2}{\pi}\right)}\left[\int_{M+\frac{1}{2}}^{\infty}\exp\left\{-2\left(\frac{x-lD}{lD}\right)^2\right\}dx + \int_{M+\frac{1}{2}}^{M+\frac{1}{2}}(x-M)\exp\left\{-2\left(\frac{x-lD}{lD}\right)^2\right\}dx\right],$$

which becomes

$$\frac{hpql}{R} = lD\left[1-(1+M-lD)P\left(\frac{2M+3}{lD}-2\right)-(lD-M)P\left(\frac{2M+1}{lD}-2\right)\right.$$
$$\left.-\frac{lD}{2}Z\left(\frac{2M+3}{lD}-2\right)+\frac{lD}{2}Z\left(\frac{2M+1}{lD}-2\right)\right],$$

where P and Z are the integral and ordinate respectively of the normal distribution.

Using this relationship, a second chart may be drawn up for determining re-order levels when the demand is normal.

5.2.2. *Re-order levels for pit stores.* Stock levels for pit stores were chosen independently of stocks at other pits in the Area. This was done because preliminary analysis had shown the great difficulty of a complete model of the problem, both in theory and in application.

The use of the normal distribution was not needed for pit stores because the demand in the lead time is usually low.

Where the annual demand for an item is less than 50, the value of q is unity, and where the demand is more than 50, q is taken as $d_i/50$. The relationships for the boundary between M and $M+1$, as derived in section 5.2.1, become

$$0 < d_i < 50$$

$$\frac{hp}{r_1} = d_i \sum_{s=m_i+1}^{\infty} \frac{(td_i)^s}{s!} e^{-td_i},$$

$$50 < d_i$$

$$\frac{hp}{50r_1} = \sum_{s=m_i+1}^{\infty} \frac{(td_i)^s}{s!} e^{-td_i}.$$

Since t may be taken as a fixed time (3 days) for replenishing a pit from the central store, these relationships may be translated into a chart, where the re-order level depends on price and demand. Run-out costs which were used for items at pit stores were given in 4.5 and 4.7.

5.2.3. *Re-order levels for the central store.* To apply the analysis of section 5.2.1 to determine stock levels for the central store the average cost of running out there must be known. For non-portable priority items an estimate was given in 4.6; for non-priority items the estimate was given in 4.7. For portable priority items the estimate given in 4.5 needs explanation.

When the central store cannot supply, the demand may be an urgent one for a pit that has also run out, but usually the demand will be to replenish a pit store. Failure to satisfy that demand simply forces the pit store to operate at below its best level for a certain time. It is the average cost of running out at the central store that is required to determine central stock levels. It will be shown that this average cost itself depends on stock levels in the Area. This situation requires an iterative solution.

Because the average run-out cost at the central store depends on stock levels in the Area, which in turn depend on demands and prices of the items used, it had to be calculated for the whole sample of items.

The average cost of a run-out at the central store depends on the stock level because a low level causes the store to be out of stock for a long time each time that it is out of stock. The pits must then manage on reduced stocks for a long period and a high proportion of their demands on the central store will be for items which are required immediately to get production going.

The converse applies if the stock level is high. Thus a low stock level means a high average run-out cost, and vice versa.

The first step was to guess a value for R, the average run-out cost, and to determine the re-order levels for all items in the sample by the equation given in 5.2.1. These stock levels were then used to obtain a better estimate of the run-out cost as described in the Appendix. This process was repeated until no improvement was observed. The final value of R was then used to draw up re-order level charts for the central store, as mentioned in section 5.2.1.

6. RESULTS

6.1. *Presentation*

The results of the above analysis were used to draw up tables of re-order levels and order quantities for pit and central stores. In each case the value depends on price and annual demand. Variability of lead time was not incorporated in the analysis. The tables were drawn up for each of a number of values of lead time and at this stage we rely on the stores official to make his best estimate of any lead time.

The tables have been applied without any serious difficulty and to the satisfaction of the stores staff.

6.2. *Sensitivity of the Results*

In this analysis considerable faith has been placed in the estimates of run-out costs and the various averaging processes. Stock levels for the sample were recalculated on the assumption that estimates of run-out cost were wrong by a factor of 2. The effect on stocks is given below:

Relative size of run-out cost	Pit stocks	Central stocks	Total stocks
$2r$	118%	113%	114%
r	100%	100%	100%
$r/2$	90%	89%	90%

This result indicates that the final rules are insensitive to the estimates of run-out costs. The results give widely differing run-out probabilities for different items. We therefore disagree with the principle of grouping items by importance and then assigning a fixed run-out *probability* to each group. It is more important to assign a run-out *cost* of the correct order to each group.

ACKNOWLEDGEMENTS

The authors wish to thank the local staff for their assistance and technical advice on this problem. They would also like to thank the National Coal Board for permission to publish this paper. The views expressed in it are their own, and not necessarily those of the National Coal Board.

APPENDIX

Estimation of Average Run-out Cost at the Central Store for Portable Priority Items

First let l^* be the average length of time that the central store is out of stock and has at least one demand outstanding; then the average time that the pits have to manage without being reprovisioned is given by

$$l' = l^* + t.$$

l^* is found by taking the expected total time during the lead time that the central store is out of stock and has an outstanding demand, and dividing it by the probability of attaining this state during the lead time. This is given by

$$l^* = \frac{\int_0^l (l-x)\frac{(xD)^M}{M!}e^{-xD}\,D\,dx}{\sum_{s=M+1}^{\infty}\frac{(lD)^s}{s!}e^{-lD}}$$

$$= l\left\{1 - \frac{\frac{M+1}{lD}\sum_{s=M+2}^{\infty}\frac{(lD)^s}{s!}e^{-lD}}{\sum_{s=M+1}^{\infty}\frac{(lD)^s}{s!}e^{-lD}}\right\}.$$

This is true for items with Poisson demand, in which class were all priority items in the sample.

If there are w pits using an item and pit "i" is the first to need replenishing when the central store is out of stock, then the expected cost for demands that the pit cannot meet in the period l' is approximately

$$E_i = r_2 \sum_{s=m_i+1}^{\infty}\frac{(l'd_i)^s}{s!}(s-m_i)\,e^{-l'd_i}.$$

The other pits all have stocks of more than their re-order levels, and have to last for time l' before being replenished. The expected cost of this is approximately

$$F_i = r_2 \sum_{\substack{j=1\\ j\neq i}}^{w}\sum_{s=m_j+2}^{\infty}\frac{(l'd_j)^s}{s!}(s-m_j-1)\,e^{-l'd_j}.$$

In normal circumstances pit "i" would have run down to its re-order level approximately $(1+l'd_i)$ times, which would have cost

$$G_i = (1+l'd_i)\,r_1 \sum_{s=m_i+1}^{\infty}\frac{(td_i)^s}{s!}(s-m_i)\,e^{-td_i}.$$

Again, in normal circumstances, pit "j" ($j \neq i$) would have run down to its re-order level approximately $l'd_j$ times, which would have cost

$$H_i = r_1 \sum_{\substack{j=1 \\ j \neq i}}^{w} l'd_j \sum_{s=m_j+1}^{\infty} \frac{(td_j)^s}{s!}(s-m_j)\,e^{-td_j}.$$

The situation which we have described enforces an extra average saving in stock holding costs at the pits of

$$J = ph(l'-t)\left\{1+\frac{D}{2}(l'-t)\right\}.$$

The average extra cost of pit "i" not being replenished and the Area having to last for l' without extra supplies is given by

$$\begin{aligned} C_i &= E_i + F_i - G_i - H_i - J \\ &= r_2 \sum_{j=1}^{w} \sum_{s=m_j+2}^{\infty} \frac{(l'd_j)^s}{s!}(s-m_j-1)\,e^{-l'd_j} \\ &\quad + r_2 \sum_{s=m_i+1}^{\infty} \frac{(l'd_i)^s}{s!} e^{-l'd_i} - r_1 \sum_{j=1}^{w} l'd_j \sum_{s=m_j+1}^{\infty} \frac{(td_j)^s}{s!}(s-m_j)\,e^{-td} \\ &\quad - r_1 \sum_{s=m_i+1}^{\infty} \frac{(td_i)^s}{s!}(s-m_i)\,e^{-td_i} - ph(l'-t)\left\{1+\frac{D}{2}(l'-t)\right\}. \end{aligned}$$

We must average this over all pits for each item:

$$\begin{aligned} C &= \frac{1}{D}\sum_{i=1}^{w} d_i C_i \\ &= r_2 \sum_{i=1}^{w} d_i\left(\frac{1}{D}+l'\right) \sum_{s=m_i+1}^{\infty} \frac{(l'd_j)^s}{s!} e^{-l'd_i} - r_2 \sum_{i=1}^{w} (m_i+1) \sum_{s=m_i+2}^{\infty} \frac{(l'd_i)^s}{s!} e^{-l'd_i} \\ &\quad - r_1 \sum_{i=1}^{w} d_i\left(\frac{1}{D}+l'\right)\left\{td_i \sum_{s=m_i}^{\infty} \frac{(td_i)^s}{s!} e^{-td_i} - m_i \sum_{s=m_i+1}^{\infty} \frac{(td_i)^s}{s!} e^{-td_i}\right\} \\ &\quad - ph(l'-t)\left\{1+\frac{D}{2}(l'-t)\right\}. \end{aligned}$$

This is now a fairly simple form for computation since all the functions are tabulated.

The average number of run-outs at the central store in the period l' is $1+l'D$. The average cost of a run-out is

$$R = \frac{C}{1+l'd}.$$

Letters to the Editor

THERE are three points with regard to my paper, Vol. 12, No. 2, June 1961 of the *Operational Research Quarterly*, "The Application of the Negative Binomial Distribution to Stock Control Problems", which I think will interest your readers.

The first is that Equation (2) may be proved by noticing that both sides of the relationship express the probability of achieving less than k successes in $n+k$ Bernoulli trials. This was brought to my notice by Mr. R. Thatcher of the Admiralty who pointed out that the relationship was derived in this way by De Moivre in "The Doctrine of Chances", published in 1756!

The second point is that numerical trials have convinced me that the straightforward normal approximation to the negative binomial distribution (which I rather unkindly called "crude") is generally to be preferred to approximation (5).

Finally I should like to draw attention to the fact that my "black box" model can be applied to a chain of k single-server queues in series each with the same negative exponential service time distribution (the service times being independent) and traffic intensity $q = \lambda/\lambda+\mu$ (cf. Jackson, R. R. P. 1956—*J. Roy. Stat. Soc.* B, **18**, 132). This fact should be of more than academic interest to those applying the model to the problem of provisioning rotable components. This is because in this case the two complementary models may well assist in the elucidation of the relationship which holds in practice between the three major factors—float, overhaul facilities and shortage risk. Progress in this direction is an important step towards obtaining an optimum economic balance between the three factors which appears to me to be the ideal ultimate aim in this field.

C. J. TAYLOR

39, Hounslow Road,
Feltham, Middlesex.

Dear Sir,

THE following quotation appeared in an article by Hua To-Keng in "China Reconstructs", August 1961:

"In promoting 'operations research', we have made full use of such means as broadcasting, television, ballad singing to the accompaniment of bamboo clappers, story telling, graphic exhibitions, lantern slides and other ways. Since the Han language is mono-syllabic and lends itself easily to snappy rhymed couplets, we explore this traditional characteristic to the full, putting down the principles in catchy verses so that they can easily be grasped and remembered."

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May I suggest something on the following lines for English usage.

A CHILD'S GARDEN OF OPERATIONAL RESEARCH

The Grand Old Duke of York, they say,
He had ten thousand men
He called in O.R. experts, Sir,
And now he's only ten.
For on a closer study, it was readily apparent
He didn't need those men at all
He needed a deterrent.

You've heard the tragic story, Sir,
Of dear old Mother Hubbard
She went to fetch her dog a bone
But found an empty cupboard
She called in O.R. experts to allay his discontent,
And now he gets a service level
Of ninety-nine per cent.

You've heard of the Mad Hatter, Sir,
And of his famous tea
He had a caterer's problem there
And so we used L.P.
We've kept his daily diet down, to milk and meter*cal*
His teas are not so sumptuous now,
But, Sir, they're opti*mal*.

Jack Horner in his corner, Sir,
Was not so very dumb
He put his finger in a pie
And pulled out a great plum
Success has made him subject
Of legend and of fables
He didn't leave it all to chance
He used our Poisson Tables.

You've heard of Humpty Dumpty, Sir,
He fell right off his wall
If you're asking our opinion, Sir,
It was a needless fall
If he'd had Ergodic Theory, Sir,
Beneath his shining pate
He'd have known before he went up there
It was a transient state.

B. GERSHILL

Operational Research Department,
B.I.S.R.A., London, S.W.11.

Book Reviews

Probability: A First Course.

F. MOSTELLER, R. E. K. ROURKE, and G. B. THOMAS, JR.

Reading, Mass., and London: Addison-Wesley. xv + 319 *pp.* £1 10*s.*

Probability with Statistical Applications.

F. MOSTELLER, R. E. K. ROURKE, and G. B. THOMAS, JR.

Reading, Mass., and London: Addison-Wesley. xv + 478 *pp.* £1 18*s.*

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In recent years, the teaching of mathematics in the U.S.A. has undergone extensive revisions. These two books mark the extension of the revolution to the teaching of statistics. One startling aspect of the new teaching is that Americans now learn set theory before calculus. This makes a formal approach to probability theory possible even in elementary books. The absence of calculus is less of a handicap than one might imagine since least squares is its main application in introductory courses and the square to be minimized can be shown to be a sum of squares in which some terms must vanish at the minimum.

The first book is an abbreviated version of the second one using the same type setting. The chapter headings are: 1. Probability and Statistics. The Study of Variability. 2. Permutations, Combinations and the Binomial Theorem. 3. A First Look at Probability: Equally Likely Outcomes. 4. General Theory of Probability for Finite Sample Spaces. 5. Numbers Determined by Experiments. Random Variables. 6. *Joint Distributions and Continuous Distributions. 7. Repeated Trials with Two Types of Outcomes: the Binomial Distribution. 8. Some Statistical Applications of Probability. 9. *Theory of Sampling. Variances of Sums and of Averages. 10. *Least Squares, Curve-fitting and Regression. The starred chapters are omitted in the first book.

It will be evident by comparison of these topics with the length of the books that the pace is leisurely in comparison to that of most elementary books. The t, F and χ^2 distributions are not described at all. There is, however, a discussion of significance testing in which the reviewer was pleased to note a careful distinction between one-sided and two-sided tests.

Difficulties and disagreements about the interpretation of probability are discussed in the introduction and the reader is advised to postpone his identification with any particular school of thought. The second book has a section on Bayesian inference and not all the examples set would be regarded as legitimate uses of Bayes' theory by everybody.

The text is full of examples and there are many exercises for the reader. The answers to even-numbered exercises are given in the second book only.

These are good books and teachers of first courses in statistics have cause to be grateful to the authors. They are, however, intended for formal courses to young students and mature students will expect to reach, more quickly, methods useful to their work.

However, those of us who have to teach mature students will find many new ideas on presentation. I am unaware of any other elementary book, for example, which derives the formulae for the mean and standard deviation of the binomial distribution from the fact a binomial variable can be regarded as the sum of a number of random variables which take the values 0 or 1.

My complaints are few and mainly trivial. The definition of the sample variance as $s^2 = 1/n\Sigma(x_i - \bar{x})^2 n_i$ can lead to difficulties when the student reaches the t-test. Probabilities are converted to odds in favour instead of the more usual odds against. The correlation coefficient is called the correlation. There is a misprint at the bottom of page 368.

The examples are chosen, of course, for the American student. However, it should not be impossible for the teacher to devise examples based on cricket instead of baseball!

The rise of a generation with such careful training in probability and statistics can only be a good augury for the quality of Operational Research in the future.

F. BENSON

The Institute of Actuaries Students' Society—The First Fifty Years 1910–1960.

F. A. A. MENZLER, C.B.E.

The Institute of Actuaries Students' Society, London 1960. x+166 *pp.* 25*s.*

This book is not about Operational Research which receives just a passing reference or two as the subject of lectures by guests. It is, however, a book about kindred spirits; about a young professionalism trying to establish itself; about the struggle to set and maintain standards without being dreary and stagnant; about the conflict between the practical and the academic, epitomized in the mock trial of the Journal Editor, Seal, on a charge of "fraudulently converting the sum of £500 or thereabouts to a purpose other than that for which it was entrusted to him, namely, by using it to edit and publish a highly abstruse mathematical and statistical periodical which was of no interest or value to the members of the said Society".

This story is unfolded by one who lived through it all, with obvious attention to historical fact and nicely restrained drama. It surely merits attention from all those concerned to guide the development of a new branch of learning.

R. T. EDDISON

Models and Analogues in Biology. Symposia of the Society for Experimental Biology.

The University Press, Cambridge, 1960. 255 *pp.* 50*s.*

In 1925 Lotka published his great essay on biological processes which he called "The Elements of Physical Biology". This he divided into four distinct but integrated parts, in the first of which he defined his approach to evolutionary factors, in the second he demonstrated the role of biologically evolving systems, the third—and perhaps the most difficult—was a discussion of the theory of statics and its place in the general principles of equilibrium; and the fourth part was devoted to dynamics and what he called the "energy transformers of nature". This in 1925 was both a scientific and literary *tour de force* for one man.

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Now, thirty-six years later, we might profitably regard this symposium on Models and Analogues in Biology as an extension of the ideas and theories brought together by Lotka, and consider the advances that have been made during these years. First, let it be said that all the old problems are still here, but this time they have new names. The stage for this modern review is set for us by a physicist, Professor Niels Bohr, who outlines some possible biological dividends from nuclear physics. There follow then four pages on genetics, kinetics, cytology and muscle structure. The next two papers on physical and mechanical models are interesting and representative of work being done by the Cambridge school.

But, with the exception of the paper on bird flight, the themes of the next six papers are central to those which face every operational research worker, whatever his original training. This group contains a most provocative and important paper on the estimation of parameters, a modern assessment and interpretation of what Lotka considered as "the parameters of state" (Chapter 23). With this paper and those which follow on electrical analogues, computers and the nervous system, cybernetics, nervous activity and energy, we are in the thick of the fray. For now we are brought face to face with one of the most difficult developmental stages of Operational Research, the translation from the quantitative description and symbolic formulation of the problem into the working model. These are the most difficult papers of the whole symposium, and they must be read *in toto*, if we are to understand, even partly, the fascinating complexities of the order-disorder phenomena of the life processes.

The review section at the end by Professor J. E. Harris, F.R.S., of Bristol, constitutes an excellent critique of the uses—and possible abuses—of the scientific approach in biology.

The need for such a scientific approach to contemporary problems in the whole field of biological endeavour must surely be one of the biggest challenges to operational research workers all over the world—for the prize is man's survival.

Here then are no ordinary games against Chance, but the whole of nature itself, where events are always Compound, where the outcomes are ever in Conflict and whose strategy is never Complete!

JOHN MAITLAND

Equitable Payment.

ELLIOTT JAQUES.

Heinemann. 336 *pp.* 35*s.*

Dr. Jaques develops theories on the related matters of the measurement of "levels of work" and the individual's "rate of consumption".

A distinction is drawn between current systems of "job evaluation" and the measurement of level of work based on the "time-span of discretion" which by quantifying responsibility is claimed to constitute a yardstick for measuring purposes. The time-span of discretion is defined as "the period of time during which marginally sub-standard discretion could be exercised in a role before information about the accumulating sub-standard work would become available to the manager in charge of the role".

To accept this theory as the basis on which pay differentials are calculated necessitates the dismissal of the various methods of job evaluation which endeavour to take into account such matters as skill (measured in training time), integrity, physical conditions and so on. Some might find this difficult. The conclusion that men in charge of large and small ships (or factories) should expect the same reward would probably be unacceptable to many.

The author discusses the matter of supplemental payment for scarcity value in the case of technologists and skilled personnel and suggests (with reinforcement from an example) that cases where there is an apparent need to pay a higher rate than the job appears to justify may well be due to an under-estimate of the level of responsibility resulting from the absence of the means of measurement. While this may be true, scarcities appear to exist in some areas (e.g. typists) and more precise measurement of responsibility can hardly be expected to disprove them.

Considerable research has been carried out into earning data of some 250 persons employed by six companies and a clear pattern is shown of the growth of individual capacity during the working life. Data sheets for income levels up to £10,000 per annum can be obtained from the publishers. It might be assumed that, in the light of these charts, which are based on past history, payment on the whole has been surprisingly equitable.

Accepting that an individual's capacity to work can be measured by the time span of discretion, Dr. Jaques postulates that this is related to the same individual's capacity for discriminating expenditure: ". . . the level of a man's economic needs can be stated in terms of capacity and, so stated, can be

quantified, his capacity to spend and his level of satisfaction consumption being commensurate with his capacity to produce". This hypothesis, the author informs us, is worth testing.

L. H. S'J. YOUNG

Glossary of Terms in Computers and Data Processing.

BERKELEY and LOVETT.

Berkeley Enterprises, Inc., Newtonville, Mass. vi+90 *pp.* $3.95.

Being the fifth edition of the "Computers and Automation" Glossary, this book has the status of its pedigree amongst the dozens of present-day glossaries and dictionaries that deal with Automatic Data Processing. It does not, as some do, throw in another subject to bulk out the volume without the inclusion of less well-known terms; with over 1000 terms (including synonyms) explained, it agrees in scope remarkably well with other serious publications of its type. The definitions amount to about 35,000 words in all, which is about average for the number of terms. On an overall view, therefore, it has few faults, except that, in the reviewer's opinion, alphabetical order is inferior to subject classification plus an index.

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Most of the definitions are very helpful to a reader who wants unfamiliar words explained to him, and nothing really misleading has been found. The glossary, however, would be very little use for verifying that one has selected precisely the correct term for the purpose in mind. The definitions are, for the most part, too loosely worded to permit critical comparison of cognate terms. Finally, a quotation can be given of a definition that is reasonably typical:

> "Operations research—The application of scientific and mathematical methods, techniques, and tools to problems dealing with the operations, procedures, and equipment of a system so as to provide those in control of the system with optimum solutions to problems."

G. C. TOOTILL

Mathematical Programming.

S. VAJDA.

Addison-Wesley Publishing Co., Inc., 1961. 310 *pp.* 68*s.*

This book aims to provide firstly a textbook of Linear and Non-linear Programming and secondly a guide to more recent development in this field.

The first six chapters give a complete account of the theoretical side of Linear Programming. Together with two chapters on Economic Interpretation

and Selected Applications, this section of the book covers, with greater vigour and depth, topics included in the author's previous slim volumes: as such it provides an excellent introduction to the subject. It is significant that the Theory of Games, which figured largely in Dr. Vajda's previous books, is here disposed of as one of the selected applications.

The last four chapters introduce the topics of Discrete Linear Programming, Stochastic Linear Programming, Non-linear Programming and Dynamic Programming. It is these chapters which give a new and more significant flavour, and make the book of interest to the practitioner as well as to the student. The emphasis is still on the theoretical aspects of the subjects; specific algorithms are outlined, and references given to an extensive Bibliography.

These advanced topics are under rapid development and the selection of material for this section is inevitably influenced by the author's own interests. Dr. Vajda disclaims any intention to cover the computational aspects of Mathematical Programming, but it is a pity that this leads him to ignore even the theoretical aspects of solving large-scale problems. Partitioning methods for Linear Programming problems are of considerable theoretical interest in addition to their practical importance, and an account of the Wolfe-Dantzig Decomposition principle, for example, would be a valuable addition to the book.

Dr. Vajda's style is admirably clear, and the subject matter has been well arranged. Only fairly elementary mathematical knowledge has been assumed, and an appendix on matrix theory has been included. A useful selection of exercises is given at the end of each chapter, and a full set of solutions is provided.

This book can be thoroughly recommended to the student of Mathematical Programming; for those with experience in this field, its usefulness is probably confined to the later chapters.

A. R. CATCHPOLE

Mathematics of Engineering Systems.

DEREK F. LAWDEN.

University Paperbacks, Methuen & Co. Ltd., 1959. viii+404 *pp.* 12*s.* 6*d.*

The brief review of this book is that it is a textbook of ordinary differential equations, linear and non-linear; that it is written for students of electronics, telecommunications, and servo-mechanisms; that the development and explanation seem very easy and clear; and that its interest to the Operational Research worker is that it gives a simple introduction to Laplace and Fourier transforms as techniques for solving D.E.'s, and has a short section on difference equations. It is not a book to replace any you may have already, but a very useful one to buy if you have no other covering these subjects.

The author, who is Professor of Mathematics at the University of Canterbury, New Zealand, has based the text on courses of lectures that he gave at the Royal Military College of Science and the Birmingham College of Technology. In the preface he states that the mathematical background assumed of the reader is approximately that of Part I, London University, B.Sc. (Engineering). The introduction, which is Chapter 1, is a preliminary discussion of analysis and revises limits, differentiation and integration, elementary functions, and complex numbers. This is a model of clarity and style, somewhat reminiscent of G. H. Hardy; its main results are used later in the book, but somehow it seems peculiarly detached from the rest of the text and not entirely adequate. There is no textbook of this level and type written specifically for the Operational Research man, and he would probably get a better basis in Defares and Sneddon, "An Introduction to the Mathematics of Medicine and Biology", though that is a much more expensive publication.

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Chapter 2 deals with the classical methods of solving linear D.E.'s with constant coefficients, including simultaneous D.E.'s, and ends with a discussion of the stability of systems. Chapter 3 introduces "modern" methods of solution, starting with the response of stable linear systems to different inputs and developing the method of the Laplace transform. This leads naturally to linear difference and differential-difference equations, but their treatment is hardly adequate for the Operational Research worker and there is no mention of probability. Chapter 4 is devoted to Fourier Analysis, and Chapter 5 to non-linear ordinary D.E.'s: these are the real "meat" of the book and reach to post-graduate level, but are naturally rather specialized in the applications though the teaching is quite general.

The book is quite excellent for its particular purpose but is only likely to be of interest to Operational Research workers in two ways: as an introductory text in differential equations and transform methods for the man who has not read them previously—Chapters 2 and 3; and for the graduate who has not solved D.E.'s for some years and wants an easy reference book to help him in particular cases. Each chapter has a brief bibliography and selected references; there is a short appendix of Laplace transforms; solutions and hints are given for the exercises in the text; there is a good index.

A. P. M. PURDON

Analysis for Production Management.

BOWMAN and FETTER.

Richard D. Irwin Inc., Homewood, Illinois, 1961. xiii + 535 *pp.* (*also Appendices and Index*). $11.65.

What the authors have done here is to compile a useful selection of analytical tools and management aids whose unifying principle is a common risk of being insufficiently widely used by industry simply because they do not as yet

form established features of the training for the professions generally represented on industrial staffs.

The work will have its value as a textbook for the appropriate readership, which the reviewer expects to include students in engineering and in the more serious management courses. It will be less satisfactory for the "interested industrialist" (Preface, p. vii) trying to read it unaided. It claims no great mathematical rigour and indeed one of its virtues is to make accessible to a wider public certain ideas which, although no longer novel, are still too often presented in language intelligible only to the few with suitable training. Its level of sophistication, however, varies.

Contrary to the authors' intention, the book falls into divisions not by progressive difficulty, but by topic, at least after section 1 which contains both introductory chapters and one chapter, purely descriptive, about management charts and diagrams. Section 2, on mathematical programming, stands largely on its own and stresses techniques rather than theorems. It is not required for section 3, which deals lucidly with quality control, and rather abruptly with analysis of variance and significance tests. Neither section is a necessary preparation for section 4. Of this, two chapters consist of a series of models culled from queue and inventory theory together with some tailor-made for cases in the authors' industrial experience. The division of these models between chapters entitled "Total Value Analysis" and "Incremental Analysis" seems artificial and could be misleading. The authors seem to be inspired by the economists' marginal analysis without noticing that with economists "the margin" enters into "total" analysis as a stage in it, not as an alternative to it. One of their examples of "Incremental Analysis" is the same as a problem already solved more elegantly in the preceding "Total Analysis". Statisticians would have found more meaningful a division between discrete and continuous models. The third chapter, on simulation, is evidently aimed at readers without computers and is accordingly cautious in its indication of the scope of such techniques. The fourth deals with renewal policy for equipment which changes in profitability through time. It is valuable in focusing attention on the assumptions implicit in renewal decisions. Full justice is done to the practical difficulty in estimating the quantities and specifying the functions assumed "known" in such models and trouble is taken to explore the sensitivity of solutions to errors of estimate.

As a textbook, this work invites a few pedagogical criticisms. Chapter 2, on the setting up, testing and use of models, although admirable in its intention, will be lost on the student unaided by a teacher if he is not already familiar with model-building. He may also have difficulty in adapting his new knowledge to the solution of some of the numerous, and otherwise valuable, "Problems" with which chapters terminate. These are clearly intended, like the more elaborate "Cases" forming the last quarter of the book, to be hammered out by groups in class. He would also appreciate more consistent notation

at some points. For example the relationship of the tables on pp. 302–3 with the text, p. 292, is obscured by the use of both "n" and "N" for the same variable. Figure 54 is bedevilled by the use of "n" (for channels) where "r" (and earlier "c"), was used in the text. In pp. 299–301, after the equation $dP_n(t)/dt = 0$ the P_n are no longer functions of time and it is confusing to continue to write $P_n(t)$.

On a point of pedagogy again, is it desirable to invite estimation of the standard deviation (for setting quality control lines) on p. 171 with a simple average of the sub-sample standard deviations? If the student has gone to the trouble of computing twenty of these, he may as well go on to use his sums of squares to compute the correct within-sample variance.

In l. 8, p. 291, the words "in operation" should be deleted and the lower summation limit changed to $n = c+1$ in eq. 8.

The authors may welcome notice of some errors which have crept into this edition.

On p. 18, "Bay(e)s'" theorem requires an "e".

On p. 26, "A" is misleadingly given two meanings, increasing the difficulty of an already awkward appendix.

On p. 297, line 18, "+..." is omitted.

On p. 297, eq. (7) should have an approximation sign.

On p. 297, lines 21, 22. Has a line of print been lost?

Page 300, equations (18) and (19). The expression given for "expected waiting time", $E(w)$, is in fact that for "expected number in queue", $E(q)$. $E(w)$ would be $A/S(S-A)$.

Page 301, line 4. $n > r$ should be $n \geqslant r$.

Page 301, eq. (22). The same error as in eqs. (18), (19) above occurs.

Page 303. Reference to table 54 plotting "mean length of line, $E(n)$". The table appears to be plotting mean waiting time (as a multiple of mean service time), that is, in the authors' notation, $E(w) \div 1/S$.

Page 328, first equation. The lower summation limit should be $c = 0$, not N.

A. D. AIRTH

Quantitative Analysis for Business Decisions.

BIERMAN, FOURAKER, and JAEDICKE.

Richard D. Irwin, Inc., Homewood, Illinois, 1961. xi+329 *pp.* (*also Appendices and Index*).

This is a further recruit to the growing army of textbooks covering some or all of the now familiar topics—basic probability and some common distributions, inventory control, simplex and transportation methods, investment and replacement models, queues, game theory, simulation—to which authors add a few chapters, reflecting their main interests, and connective tissue. The present

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example is better integrated than might be expected from multiple authorship. The authors do not, nevertheless, suppose that all readers will work from cover to cover.

After a brief but good chapter on model making, the book launches out lucidly into probability, following fairly closely on Schlaifer's "Probability and Statistics for Business Decisions" to which due homage is done. The subsequent treatment of the Normal Distribution will seem abrupt to the reader unfamiliar with statistics, especially in view of the authors' subsequent heavy use of Normality as an assumption. Trying to choose "the simplest means of exposition" and to avoid "proofs and much of the characteristic rigour" of other treatments, the authors have not, as they hoped, retained "the essential subtlety of the techniques", but have exposed the less discriminating student to the danger of using the Normal distribution much as the school boy regards logarithms to 4 places, as something to be invoked whenever the situation goes beyond the capacity of ten fingers. This danger is not averted by the isolated concession, such as "despite its general application it should not be assumed that every process can be described as having a normal distribution" (p. 43), since the authors proceed virtually as if it could.

Elementary stock models are lucidly portrayed. So is the two-person zero-sum game, apart from an inconsistency in the use of the utility concept of decision theory (pp. 91 and 95).

A potentially useful chapter applying quality control concepts to cost control wastes space on elaborating limits based on the quite fortuitous assumption of normally distributed costs, to the virtual neglect of the essential division between controllable and uncontrollable deviations.

Chapters 10 and 11, on investment in stocks and on capital budgeting respectively, are exercises in the concepts of utility and decision-making under uncertainty. They come close to elaborating on the obvious, since the real problems are eliminated by assumption. Sometimes the assumptions are surprising, to say the least, as when the price and the dividend of a security are supposed independent random variables, and the utility functions of individuals are assumed "known".

Much more satisfactory is Chapter 12, a careful elaboration of an example of Schlaifer's (*op. cit.*) on decisions involving "buying" information by taking samples, and on the subsequent revision of prior probabilities. It covers much ground without loss of lucidity, and should serve its purpose in inciting the newcomer to statistics to pursue the topic further.

Chapter 13 handles in 17½ pages "Electronic Data Processing, Simulation, and the waiting line problem", taking in the Poisson and exponential distributions on the way. The authors' ingenuity in achieving so much commands respect, but one may be forgiven for asking "Aren't we a little ambitious?"

Sixty pages, in three chapters, on the other hand, are devoted to Linear Programming. Given this space in which to work the authors achieve a good

exposition of basic techniques, relating carefully the routine computations to the algebra of more formal treatments. One might ask, nevertheless, for a less hasty word on degeneracy.

Four chapters follow which should have formed an important part of the book, and would do so if this were primarily a mind-training book, not one concerned with "techniques which are useful in business decision-making" (*vide* dust-cover). They may yet be redeemed in a later edition if the authors find a means of reducing their dependence on assumptions chosen, not as good working approximations to reality, but because the answers come out more easily that way. These chapters contain, the reviewer imagines, much of the authors' original contributions to the work. Some errors may thus be forgiven. Nevertheless, aiming at students rather than colleagues, the authors owe the reader more careful presentation.

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Chapter 22 is a good attempt to portray information theory in 9 pages. Isolated as it is both from applications and from other chapters, however, its inclusion serves no obvious purpose.

A short chapter on replacement theory is presented as an exercise in calculus.

A work such as this, crossing the boundaries of several disciplines, presents the student with many new terms. Clear statements of definitions, and a more detailed index to locate them, would increase the value of the book. The examples are well chosen to exercise the student in the material of the text even if he is unaided by a teacher.

Perhaps a later edition will correct the linguistic and printing errors which have escaped the proof-reader.

A. D. AIRTH

The Clerical Function.

ALLAN A. MURDOCH and J. RODNEY DALE.

Pitman, London, 1961. 205 *pp.* 30*s.*

This book was designed mainly for students and written specifically to cover the syllabus for "Office Methods" in the Diploma for the Institute of Office Management. It contains a lot of useful material, but is too small in compass to justify the claim made on the dust-jacket that the subject-matter is given comprehensive treatment. The arrangement of the contents and its presentation would be unlikely to satisfy the operational research worker seeking a concise and lucid introduction to contemporary office systems and methods.

The three opening chapters describe some typical clerical and accounting systems, and also refer to, and explain, certain office machines and equipment. Next follows a chapter outlining the use of work study in the office. The final four chapters contain a general survey of methods, processes, machines and equipment used in the office which in places overlaps earlier references to clerical aids.

In the main the text is readable, but occasionally irritatingly circumlocutory. The technical descriptions of office machines tend to be elementary and obscure. Some questionable views are expressed regarding trends in electronic computer development and, in this day and age, it is surprising to be told that the adoption of computers for commercial purposes is "as yet, only in the earliest experimental stage". A large number of photographs and line diagrams are included. Not all of the latter can be easily reconciled with the text, and in a book purporting to survey modern office methods it is curious to find a plate devoted to illustrating an early typewriter *circa* 1875.

B. M. BROUGH

Progress in Operations Research.

R. L. ACKOFF, *Editor.*

John Wiley, New York. Vol. 1, ix+491 *pp.*

This book is No. 5 in the series of "Publications in Operations Research" sponsored by the Operations Research Society of America. The addition of a volume number to the title shows that it is the precursor of a number of review publications. No promise is made, however, that these reviews will appear regularly. Their appearance depends on the rate of progress and on the reception accorded to this volume.

A teacher has two possible tasks when requested to review a book. The book may deal with material with which he is thoroughly familiar and his job is then to decide whether it adds significantly to teaching methods. In other cases, he may be faced with a book which enlarges his own thinking to such an extent that he is tempted to plead for time to read the book through, a three months' gap, then to re-read the book and spend six months digesting and thinking out its lessons before committing himself to print. This book falls into the second category but the reviewer was not granted his wishes.

The avowed purpose of the book is to "evaluate the current state of the art, survey important applications and discuss deficiencies and unsolved problems". This purpose is so vital that, if it is justified, we should all run to the nearest bookseller. With the exception of one chapter, the claim is justified. Moreover, it is not a book to borrow for two weeks from a library.

The chapters fall into two main groups—philosophical and technical. The philosophical chapters are the ones which demand reading and re-reading. They concern the very nature of Operational Research and we neglect them at our peril. The other chapters survey the state of knowledge in the traditional fields—inventory, mathematical programming, dynamic programming, Markov and queueing processes, sequencing, replacement, simulation, military gaming. These chapters fulfil the purpose of the book with the exception of the chapter on queueing. This is the maverick in the book—one has a depressing sense of having seen it all before, even if the language has been changed. To be carping,

the book as a whole could be criticized for the paucity of non-American references—in queueing theory, this is tragic—there is no mention of Khintchine, Cox or Naor, to mention substantial contributors from three countries. The new *International Abstracts Journal* will do much to make us aware of work done elsewhere.

A book which contains chapters by many authors, most of whom have done a magnificent job, is an unusual event. The reviewer hopes that subsequent editors in the series will find equal co-operation. No Operational Research department can afford not to have this book.

H. R. W. WATKINS

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Abstracts

Elementary Theory of Optimum Storage Design.

E. SCHLIFER and P. NAOR.

Operational Research Quarterly, **12**, 1, pp. 54–65, May (1961).

This study indicates a method for calculating an economic division of warehouses into compartments when different varieties have to be stored simultaneously and separately. The study deals with cases where the total storage capacity demand and the number of varieties are known and fixed, but where the storage capacity demand of each variety is unknown. Problems of this nature exist in the preliminary designing of grain silos, fleets of transport vehicles, utensils in public kitchens, etc. If the warehouse capacity can cope with the total storage capacity demand, but the number of compartments is insufficient, it is possible that part of the material to be stored cannot be accommodated, although certain compartments may only be partly filled; this will be due to the impossibility of storing different varieties together. The materials will have to be stored in other warehouses, at a higher charge per capacity unit than that of the warehouse in question.

The division of the warehouse into a large number of compartments will assure storage of most, if not all, the material. On the other hand, increase in the number of compartments (and installations) makes for increased investment in transport installation, partitions and additional storage capacity to compensate for space taken up by these partitions and installations.

The object of this study is to introduce a method for calculating the optimal division of a warehouse into compartments, so that the annual costs of storage and capital investment are minimized.

It is possible—by introducing not very restrictive statistical assumptions—to calculate the annual costs of storing excluded material elsewhere for different modes of partitioning a warehouse into compartments. The annual building cost of the warehouse is calculated in the ordinary way.

In this study a solution is offered for the partitioning of a warehouse into equal compartments. It is to be expected that the extension of this solution to cases of the partition into compartments, differing from each other as to storage capacity, will tend to lower costs.

T. E. E.

Optimum Lot-Size Policy if Tools break down frequently.

A. R. W. MUYEN.

Operational Research Quarterly, **12**, 1, pp. 41–53, May (1961).

A well-known formula gives the optimum length of a production series by balancing change-over costs against inventory costs. This formula can no

longer be considered as optimum in the case of a high breakdown rate of the tools. We can often reduce our costs in such situations by ending the production series when a breakdown occurs. This introduces a variability in the lot-sizes, which in turn causes higher stock levels.

A production policy was developed, based on two limits, Q_1 and Q_2. Production is stopped either at the first breakdown after Q_1 units have been produced, or at Q_2. Optimum values of Q_1 and Q_2 were derived and can be read off from a set of graphs.

One assumption of this theory is that tools are always available, which theoretically requires an infinite stock of these tools.

A Monte Carlo programme was set up to study the situation, in which the number of tools was limited. The number of tools, and the reorder-level for the products, required for a desired probability of stock-out can be derived by the programme.

The theoretical values of Q_1 and Q_2 could be considered optimum for practical purposes in the cases under consideration.

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A Multiple Reorder Point Inventory Policy.

E. KOENIGSBERG.

Operational Research Quarterly, **12**, 1, pp. 27–40, May (1961).

This paper discusses the development and application of a multiple reorder inventory policy which can be stated as follows: reorder an optimal lot size Q when inventory (stock on hand) falls to R, $R-Q$, $R-2Q$, ..., $R-NQ$; where R is the reorder level. If demands cause the inventory to fall below two reorder levels, say a jump from $R+\epsilon$ to $R-2Q+\epsilon'$ where ϵ and $\epsilon' < Q$, an order for $2Q$ is placed. The policy is a form of (S, q) policy where the maximum stock level $S = R+Q$. The system is of particular value in cases where the coefficient of variation of lead time demand μ_l ($\mu_l = \sigma_l/\lambda_l$) is large (say $>0{\cdot}5$) and continuous inventory records are maintained. Tables, charts and nomographs to simplify clerical tasks can be obtained quite readily. In this formulation R and Q are not independent factors as in the usual Wilson formulation, but are obtained by minimizing a single cost functional subject to the constraint of a specified risk of out-of-stock condition or a specified level of service (Galliher and Simmond, 1957; Morse *et al.*, 1959). The particular application concerns the raw material inventories of a manufacturer of metal pressings who is required to offer "immediate service". The demand distribution during the lead time closely approximates the exponential distribution, and lead times are constant for each raw material. The application of the multiple reorder policy results in a 30 to 35 per cent reduction in inventory for a 95 per cent service level. Measures of sensitivity and response are obtained, and the mean

number of shortages is expressed in closed form. The policy is compared with the Wilson policy and shown to be more "effective" in that it results in lower inventories and a smaller number of orders for the case considered.

T. E. E.

Optimum Reliability of Parallel Multicomponent Systems.

R. N. v. HEES and H. W. v.d. MEERENDONK.

Operational Research Quarterly, **12**, 1, pp. 16–26, May (1961).

For a set of m similar systems each consisting of r phases, and with n_i spare components to the ith phase, a formula is developed, on standard probability principles, for the probability that a given one of the installations will be found defective through failure of any phase, on the assumption that the ith phase has a probability n_i of being found defective alike for the installed part and the spares. Cannibalization of systems defective at one phase to supply other phases is not considered. A simple algorithm, which can be carried through either arithmetically or graphically, is then developed to find the outlay on spares, within a given cost limit, that will maximize the probability of all systems being serviceable simultaneously.

T. E. E.

Study of the Optimum Equipment and Utilization of an Underground Gas Reservoir of Limited Capacity.

R. FORT.

Operational Research Quarterly, **12**, 1, pp. 1–15, May (1961).

In order to smooth out the fluctuations of demand for gas, gas is stored in underground reservoirs at times of low demand and withdrawn during peak periods. Taking into account (a) the maximum amount of gas that would be called for during at most two years in a hundred, between any date and the "end of the winter", (b) the capital and running costs of producing plant and of reservoirs, and (c) the maximum capacities and withdrawal rates of reservoirs, the authors develop formulae for the most economical number of reservoirs and the optimum régime for using them.

T. E. E.

Application of Dynamic Programming to the Control of Stock and to the Calculation of a Maximum Stock Capacity.

E. VENTURA.

Operational Research Quarterly, **12**, 1, pp. 66–78, May (1961).

The paper considers the case of a company which fulfils orders, which have a known distribution, by shipping from a port that can be supplied cheaply by

barge or at high cost by rail, the former being subject to random disturbances whereas the latter is quick and certain. It is desired to find the optimum number of warehouses that should be built at the port to store deliveries in excess of orders. Deliveries above warehouse capacity must be stored at high cost in rented accommodation.

When the cost of all factors for the last i periods is formulated, it becomes clear that on certain realistic simplifying assumptions the total cost, for a given amount of warehouse capacity, is a linear function of the stock at the beginning of the first of these periods. It is thus possible by dynamic programming to lay down the optimum shipment and storage policy. Since only three policies for storage capacity are compared (as current/one additional warehouse built/two additional warehouses built), it is possible to do this for each policy and find the most economical.

The effect of changing various parameters is discussed.

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The Determination of the Optimal Investment Program for a Sector of Industry, by Linear Programming.

JÁNOS KORNAI.

Kozgazdasagi Szemle, **8**, pp. 570–585, May (1961).

The paper is an account of the research project for computing the optimal investment and technical development programme of the cotton weaving industry for the period of the Second Five Year Plan, using the mathematical methods of linear programming. This was the first general problem in linear programming on a larger scale to be solved with an electronic computer in Hungary.

The calculations concerned all the weaving mills of the cotton industry and considered the looms, cross winding and pirn winding machines, dressing machines and warpers. The model was construed in such a way as to yield answers to the following problems: (1) Should the existing factories be modernized, or new factories built? (2) What is to happen to the old machinery—should the old machinery continue to be operated, be modernized or dismantled? (3) What new types of machinery should be purchased, and how many of each type? The programme determines the desirable qualitative and quantitative composition of the stock of fixed goods for the industry in 1965.

The model contained 24 conditions. These came under the following main headings: (a) the external production obligations of the industry, (b) the limits of the investment resources available, (c) the conditions which ensure vertical proportionality within the industry, and (d) conditions which express a state of affairs from the previous period. The number of variables in the model was 50. The objective function prescribed the minimalization of costs.

The calculations were carried out in 17 variants. The variants differed from one another in the output volume prescribed, the interest formula used, and the magnitude of the rate of interest and foreign exchange rates.

The calculations led to conclusions which could be adopted in practice. It turned out that it would pay to give priority to the modernization of existing mills. In respect to automation it would pay to go further than originally planned by the directors of the industry. The programme showed which of the alternative types of machine it would be best to acquire. In the full analysis, the proposal worked out on the basis of mathematical programming results in a 15 per cent saving compared to the original plan that was worked out by the traditional methods.

The programming, moreover, yielded general conclusions for economics, partly in connection with the rates of interest and of exchange used in computing the economy of investments, partly in respect to the methods of planning. It has been shown that calculations of economy with the help of a programming model can be used over a wide field. (Hungarian: Economic Review.)

T. E. E.

The Role of Models in Operations Research (with discussion).

K. D. TOCHER.

Journal of the Royal Statistical Society, Series A, **124**, part 2, pp. 121–142 (1961).

A survey is made of the application of mathematical models in Operational Research work. An Operational Research investigation is itself modelled and divided into eight stages, of which model building is the second. A classification is made of the mathematical theories of Operational Research which are listed under ten headings. Several examples are given of the study of congestion and an application of dynamic programming to a problem of energy generation is also included.

B. R. A.

The Movement of Households in England.

D. V. DONNISON.

Journal of the Royal Statistical Society, Series A, **124**, part 1, pp. 60–80 (1961).

Results of a survey show that most people who move house go to a dwelling previously occupied by someone else and relinquish a dwelling into which someone else moves. The movement of households may, therefore, be regarded as a continuing flow rather than a mass of unrelated impulses. This is stated in a simple mathematical form. Four aspects of this flow are then briefly examined: the households that move, the houses they move between, the

distance and direction of movement, and the motives for movement. The findings for each confirm and complement each other and pose a number of questions for policy makers, administrators and architects.

B. R. A.

Natural Law in the Social Sciences. The inaugural address of the President (with discussion).

M. G. KENDALL.

Journal of the Royal Statistical Society, Series A, **124**, pp. 1–18 (1961).

Consideration is given to the extent to which laws of behaviour are observed in social phenomena, the nature of these laws and how they can be used. Illustrations included the distribution of incomes (Pareto's Law), of industrial capacity (Zipf's Law), of articles on Statistics (the Yule distribution), of consumer purchases (negative binomial distribution), of the frequency of wars (Poisson distribution), of strike frequency (also Poisson) and swings at general elections. These exhibit the definiteness of the pattern of behaviour which emerges from a set of acts of entirely free choice. However, the mere eliciting of a pattern is not sufficient. Not until a satisfactory model has been established can the process be fully understood or controlled in changing circumstances. But the reduction of social activity to scientific order has begun, long task though it may be.

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An Experimental Investigation of the Robustness of Certain Procedures for Estimating Means and Regression Coefficients.

D. C. CHAMPERNOWNE.

Journal of the Royal Statistical Society, Series A, **123**, pp. 398–412 (1960).

This paper describes an investigation of the misleading conclusions which may result from the uncritical use of orthodox procedures of estimation of populations means and regression coefficients, when the underlying conditions theoretically required if the procedures are to be reliable are in various ways unsatisfied. The robustness of the distribution of Student's t is found when the requirement that the distribution of the parent population should be normal is relaxed. The effect of generating the independent variates of a regression relationship by autoregressive mechanisms is also examined. This investigation was based on the use of the random number facility in a Mercury digital computer.

B. R. A.

News and Notes

COLLEGE OF AERONAUTICS

A TWO-DAY Management Appreciation Course in Operational Research will be held on 11 and 12 January, 1962. Enquiries should be addressed to The Warden, The College of Aeronautics, Cranfield, Bletchley, Bucks.

NEWS OF MEMBERS

New full members of the Society

Mr. D. A. Brace, ORbit (Operational Analysis and Research) Ltd.
Mr. J. F. Green, Smiths Aircraft Instruments, Cheltenham.
Mr. L. R. Johnson, London Transport Executive, London.
Mr. D. L. Lamberth, Director, Shepherd and Manning Ltd.
Mr. J. D. Oates, A.R.D.E., Fort Halstead.
Mr. M. C. Pearson, Decca Radar, Operational Research Division.
Mr. A. T. Pennington, The British Aluminium Co. Ltd.
Professor C. M. Shetty, Northwestern University, Evanston, Illinois.

Changes of Appointment

Mr. C. J. Davies, previously with the Austin Motor Co. Ltd., is now with I.T.D. Ltd., Wess Lane, Hall Green, Birmingham, 28.

Mr. E. A. Field, temporarily with Esso Research & Engineering Co., Madison, New Jersey, U.S.A.

Mr. A. D. J. Flowerdew, previously with Richard Thomas & Baldwins Ltd., Newport, Mon., is now with R.T. & B., Scunthorpe, Lincs.

Mr. M. W. B. Gordon, previously with Bowater-Scott Paper Corpn., Northfield, Kent, is now with C. Shippam Ltd., Chichester, Sussex.

Miss J. Heald, previously with Distington Eng. Co. Ltd., Workington, is now with The United Steel Companies Ltd., Sheffield.

Mr. D. J. Kirby, previously with National Coal Board, is now with Wiggins Teape Paper Mills Ltd., Dartford, Kent.

Dr. B. O. Koopman, previously with U.S. Operations Research Groups, London, is now with Department of Mathematics, Columbia University, U.S.A.

Mr. A. M. Lee, previously with B.E.A., Ruislip, is now with Trans-Canada Airlines, Montreal.

Dr. T. I. McGloughlin, previously with Gillette Industries Ltd., is now with Attwood Statistics Ltd., Eccleston Street, London.

Mr. S. A. Pai, previously with Steel Co. of Wales, Port Talbot, is now with I.C.I. (India) Private Ltd., Calcutta, India.

Mr. P. K. Pearson, previously with the National Coal Board, is now with Gillette Industries Ltd., Isleworth.

Mr. D. F. D. Pope, previously with Richard Thomas & Baldwins, Ltd., Ebbw Vale, is now with B.I.S.R.A., Battersea Park Road, London.

Mr. R. C. Reinitz, previously with Case Institute of Technology, Cleveland, Ohio, is now with Arthur D. Little Co., Cambridge, Mass.

Mr. B. W. Sayer, previously with Samuel Fox & Co. Ltd., near Sheffield, is now with Arthur Andersen & Co., London.

Mr. B. H. Shorrock, previously with Metropolitan Vickers Electrical Co. Ltd., Manchester, is now with A.E.I. (Manchester) Ltd., Manchester.

Mr. R. G. Soden, previously with the National Coal Board, London, is now with B.I.S.R.A., Battersea Park Road, London.

Mr. G. W. Taylor, previously with Royal Aircraft Establishment, Farnborough, is now with Associated Eurocontrol, Paris.

Mr. R. J. Taylor, previously with B.I.S.R.A., Battersea Park Road, London, is now with Associated Industrial Consultants Ltd., London.

Mr. A. V. Vincent, previously with Workington Iron & Steel Co. Ltd., Workington, is now with A. Guinness Son & Co. (Dublin) Ltd., Dublin.

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ERRATA

Operational Research Quarterly, Vol. 12, No. 2, June 1961, pp. 133, 134. The text of the abstract from *Linear Theory*, by J. R. Hicks, appearing at the foot of p. 133 should be interchanged with the text of the abstract from *Operational Research in British Transport*, by M. G. Bennett, which appears on p. 134.